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Fundamental groups of II_1 factors and equivalence relations

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Dissertation presented in partial fulfillment of the requirements for the degree of Doctor in Sciences

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Abstract

One invariant of II_1 factors and II_1 equivalence relations that has been studied extensively is the fundamental group. This notion is not related to the fundamental group of a topological space, but instead can be shown to be a subgroup of the positive real numbers. It was introduced by Murray and von Neumann for II_1 factors. Murray and von Neumann were only able to compute it for the hyperfinite II_1 factor: $\mathcal{F}(R) = \mathbb{R}^*_+$. Since then, much has been achieved, but it also became clear that calculating this fundamental group was very difficult.

Whenever you have a countable group acting freely, ergodically and measure preservingly on a standard probability space, you can construct both a II_1 equivalence relation (the orbit equivalence relation), and a II_1 factor (the group measure space construction). In this case, the fundamental group of the equivalence relation is a subgroup of the fundamental group of the II_1 factor. Many results involving these group actions show that equality holds in specific cases. However, this is not true in general. In 2006, Popa gave examples of group actions where the difference is as big as possible. In this thesis, I wil elaborate on this, giving examples where the equivalence relation can have 'arbitrary' fundamental group, whereas the associated II_1 factor has $\mathbb{R}^*_+$ as a fundamental group.

Furthermore I give examples of another interesting phenomenon, where a II_1 factor contains a Cartan subalgebra such that the fundamental group of the equivalence relation associated to this Cartan subalgebra is non-trivial, whereas the fundamental group of the original II_1 factor is $\{1\}$. Indeed, whenever a II_1 factor has a Cartan subalgebra, Feldman and Moore showed that this gives rise to a II_1 equivalence relation and a scalar 2-cocycle. In our case, this 2-cocycle is non-trivial, and hence the equivalence relation associated to the Cartan subalgebra is twisted by a 2-cocycle inside the II_1 factor.

To find such an example as the one described above, we give a class of group actions $G \curvearrowright (X, \mu)$ that admit a second Cartan subalgebra, which is not necessarily conjugate to $\mathrm{L}^\infty(X)$. These examples are based on an earlier example by Popa and Vaes.

Beknopte samenvatting

Één invariant van II_1 factoren en II_1 equivalentie relaties die veel bestudeerd wordt is de fundamentaalgroep. Deze notie is niet dezelfde als de fundamentaalgroep voor een topologische ruimte; ze is een deelgroep van de positieve reële getallen. De fundamentaalgroep werd voor II_1 factoren ingevoerd door Murray en von Neumann. In dezelfde paper toonden ze aan dat de fundamentaalgroep van de *hyperfinite* II_1 factor heel $\mathbb{R}_+^*$ is, maar ze konden van geen enkele andere II_1 factor de fundamentaalgroep berekenen. Sindsdien is er al veel vooruitgang geboekt, maar het werd ook duidelijk dat het berekenen van deze fundamentaalgroep vaak erg moeilijk was.

Wanneer je een aftelbare groep laat werken op een standaard kansruimte, dan kan je zowel een II_1 equivalentie relatie construeren (de orbiet equivalentie relatie), als een II_1 factor (de groep-maatruimte constructie). Als je zo een groepsactie hebt, dan kan je aantonen dat de fundamentaalgroep van de equivalentierelatie een deelgroep is van de fundamentaalgroep van de II_1 factor. Veel resultaten met betrekking tot deze groepsacties tonen aan dat in specifieke gevallen gelijkheid geldt. Dit is echter geen algemene waarheid: in 2006 gaf Popa voorbeelden van groepsacties waar het verschil tussen de fundamentaalgroepen het grootst mogelijke is. In deze thesis geef ik, vertrekkende van de voorbeelden van Popa, voorbeelden waar de equivalentierelatie 'willekeurige' fundamentaalgroep kan hebben, terwijl de fundamentaalgroep van de II_1 factor $\mathbb{R}_+^*$ is.

Verder geef ik voorbeelden van een andere interessante situatie, waarbij de II_1 factor een Cartan deelalgebra bevat zodat de fundamentaalgroep van de equivalentierelatie geassocieerd met deze Cartan deelalgebra niet triviaal is, terwijl de fundamentaalgroep van de originele II_1 factor $\{1\}$ is. Inderdaad, Feldman en Moore toonden aan dat wanneer een II_1 factor een Cartan deelalgebra heeft, er een II_1 equivalentierelatie en een scalaire 2-cocykel bestaan zodat de getwiste von Neumann algebra gegenereerd door deze equivalentierelatie en cocykel de originele II_1 factor geeft. In ons geval moet die 2-cocykel dus niet triviaal zijn.

Om dergelijke voorbeelden te vinden, tonen we eerst aan dat voor een bepaalde klasse van groepsacties $G \curvearrowright (X, \mu)$ er een tweede Cartan subalgebra bestaat, die niet noodzakelijk geconjugeerd is met $\mathrm{L}^\infty(X)$. Deze voorbeelden zijn gebaseerd op een eerder voorbeeld van Popa en Vaes.

Contents

Introduction

In 1936, the first paper on von Neumann algebras [MvN36] appeared. Murray and von Neumann mentioned many motivations to study these objects, amongst others the study of unitary representations, the study of unbounded operators and giving a mathematical formulation of quantum mechanics. This paper was quickly followed by five more papers, [MvN37], [vN40], [MvN43], [vN43] and [vN49], in which they continued their research on the subject. A von Neumann algebra is a unital *-subalgebra of the algebra of bounded operators on a Hilbert space that is closed for the weak operator topology. By von Neumann's bicommutant theorem, they are characterized as those unital *-subalgebras of the algebra of bounded operators on a Hilbert space that are their own bicommutant. We are only interested in separable Hilbert spaces and von Neumann algebras on separable Hilbert spaces.

Murray and von Neumann proved [vN49] that to classify von Neumann algebras it suffices to classify von Neumann algebras with trivial center, i.e. those whose center only contains scalar multiples of the identity. These von Neumann algebras are called factors. In fact they showed that every von Neumann algebra can be decomposed as a direct integral of factors.

These factors are classified in types I, II and III. A factor is said to be of type I if it is a matrix algebra or the algebra of all bounded operators on a Hilbert space. Note that this implies that the factor contains a minimal non-zero projection. From a von Neumann algebra point of view, these are considered 'trivial'. Type II factors are factors that do not have minimal projections, but do have non-zero finite projections. A finite projection is a projection p such that there are no projections $q < p$ and partial isometries v such that $vv^* = q$ and $v^*v = p$. These are subdivided in types II_1 and II_∞ in the cases where the identity is a finite resp. an infinite projection. Finally, type III factors are factors that do not contain non-zero finite projections.

In this thesis, we focus mainly on II_1 factors. The fact that they have no minimal projection and that the identity is a finite projection is equivalent with it not being type I and having a faithful normal trace $\tau : M \to \mathbb{C}$. A trace is a positive linear functional τ such that $\tau(xy) = \tau(yx)$ for all $x, y \in M$. It is called faithful if $\tau(x^*x) = 0$ implies $x = 0$. A trace is normal if it is continuous for the weak operator topology on the unit ball of M. We usually consider the trace to be normalized, i.e. $\tau(1) = 1$. One can prove that like this, the trace on a II_1 factor M is unique.

The algebra of all bounded operators on a Hilbert space is, as said before, a first example of a von Neumann algebra. However, this example is not very interesting. More exciting examples are the group von Neumann algebra associated to a group, and the group measure space construction, associated to an action of a group on a measure space. Both were introduced by Murray and von Neumann in their initial series of papers. The group von Neumann algebra is a von Neumann algebra we can associate to a discrete countable group by considering the von Neumann algebra generated by the left regular representation. It is denoted by $\mathcal{L}(G)$ for a given group G. In case all conjugacy classes for non-trivial $g \in G$ are infinite (we say the group is an ICC group), one can show that this group von Neumann algebra is a II_1 factor. This seems like a very natural way to build a von Neumann algebra from a group, but it is usually very hard to distinguish between group von Neumann algebras, and many problems on this subject remain open.

The group measure space construction is a way to build a von Neumann algebra from a non-singular action $G \overset{\alpha}{\curvearrowright} (X, \mu)$ of a countable group on a measure space. If the action is free and ergodic, the resulting von Neumann algebra will be a factor. Depending on the structure of the measure space, this construction will give you a type I factor (in the case of an atomic measure space), a type II factor (if the measure is equivalent to a σ-finite measure that is invariant under the group action) or a type III factor (if there is no such equivalent measure).

It was shown in 1955 by Singer ([Si55]) that this construction only depended on the orbit equivalence class of $G \curvearrowright (X, \mu)$. In other words, it is interesting to look at the equivalence relation generated by the orbits of such an action. In studying these countable equivalence relations, one soon translated this classification into types to the context of equivalence relations, much in the same way a group measure space construction gives rise to different types of factors. This new theory of countable equivalence relations clearly was very closely related to the original theory of von Neumann algebras. In fact, Feldman and Moore proved in [FM75] that whenever a II_1 factor contains a special kind of subalgebra called a Cartan subalgebra, this inclusion gives rise to a II_1 equivalence relation and a scalar 2-cocycle. Furthermore in formalizing the concept of these countable equivalence relations, they showed in [FM75] that any countable equivalence relation on a standard Borel space is the orbit equivalence relation of some (not necessarily free) group action on the standard Borel space.

Neither von Neumann algebras nor equivalence relations are easy to study, in that it is often difficult to decide whether two factors or equivalence relations are the same or not. For example, Dye proved in [Dy59] that all free measure preserving actions of groups with polynomial growth are mutually orbit equivalent. Connes showed in [Co76] that all II_1 factors of the form $\mathcal{L}(G)$ for an amenable ICC group G are isomorphic to the hyperfinite II_1 factor. Ornstein and Weiss showed in [OW80] that all probability measure preserving actions of amenable groups are hyperfinite. In [CFW81], Connes, Feldman and Weiss introduced the notion of an amenable II_1 equivalence relation, and show that they are all hyperfinite. However, for $n \neq m$ it is not yet known whether $\mathcal{L}(\mathbb{F}_n) \cong \mathcal{L}(\mathbb{F}_m)$, where $\mathbb{F}_n$ denotes the free group on n generators. To distinguish between II_1 factors in general (and II_1 equivalence relations), invariants were introduced.

An invariant for II_1 factors and equivalence relations

One of these invariants is the fundamental group. It was introduced by Murray and von Neumann for type II_1 factors in [MvN43]. This notion is not related to the fundamental group of a topological space, but instead is a subgroup of the positive real numbers. For a II_1 factor (M, τ), this is the following group:

$$\mathcal{F}(M) := \left\{ \frac{\tau(p)}{\tau(q)} \mid pMp \cong qMq \right\}.$$

It can be proven that this is indeed a multiplicative subgroup of $\mathbb{R}^*_+$.

Murray and von Neumann showed that the fundamental group of the hyperfinite type II_1 factor is $\mathbb{R}^*_+$ itself. They were not able to compute the invariant for any other II_1 factor. Their line “the general behavior of the above invariants - including fundamental groups - remains an open question” still has some truth today. Calculating the fundamental group of a II_1 factor is a very difficult problem.

In fact, it took almost 40 years before one proved that the fundamental group of a II_1 factor could be different from $\mathbb{R}^*_+$. The first examples of II_1 factors with ‘small’ fundamental group were given in [Co80], as Connes showed that the fundamental group of a property (T) factor is countable.

Voiculescu proved in [Vo89] that the fundamental group of $\mathcal{L}(\mathbb{F}_\infty)$ contains $\mathbb{Q}^*_+$ and Rădulescu proved in [Ra92] that the fundamental group is the whole of $\mathbb{R}^*_+$. It is not yet known whether $\mathcal{L}(\mathbb{F}_n) \cong \mathcal{L}(\mathbb{F}_m)$ for $n \neq m$, but the following dichotomy was proven by Rădulescu in [Ra94] and Dykema in [Dyk94]: either all $\mathcal{L}(\mathbb{F}_n)$ are isomorphic and their fundamental group is $\mathbb{R}^*_+$ or they are all different and their fundamental group is $\{1\}$.

The first explicit computations of fundamental groups of II_1 factors, other than $\mathbb{R}^*_+$, were established around 2002. Using his deformation-rigidity theory, Popa proved that the fundamental group of the group von Neumann algebra $\mathcal{L}(\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2)$ is trivial ([Po02]). Later he showed that any countable subgroup of $\mathbb{R}^*_+$ arises as the fundamental group of a type II_1 factor ([Po03]). Alternative constructions for these results can be found in [IPP05] and [Ho07].

In [PV08a] and [PV08c], Popa and Vaes showed that a large class of uncountable subgroups of $\mathbb{R}^*_+$ appear as the fundamental group of a type II_1 factor. This was done through an existence theorem, using a Baire category argument. In [De10], Deprez provides explicit examples of this phenomenon.

In this thesis we study the link between the fundamental group of a II_1 factor and the fundamental group of the equivalence relation associated to a Cartan inclusion in the factor. The notion of fundamental group for countable equivalence relations is the natural translation of the notion for II_1 factors. More specifically, the fundamental group of a II_1 equivalence relation $\mathcal{R}$ on a measure space (X, μ) is the following group:

$$\mathcal{F}(\mathcal{R}) := \left\{ \frac{\mu(U)}{\mu(V)} \mid U, V \subset X \text{ non-negligible}, \mathcal{R}_{|U} \cong \mathcal{R}_{|V} \right\},$$

where by $\mathcal{R}_{|U}$ we mean the restriction of the equivalence relation to the subset U of X. More or less parallel to the results mentioned above, rigidity theory for equivalence relations became more developed. In 1987, Gefter and Golodets proved in [GG87] that for $n \geq 3$, the fundamental group of the orbit equivalence relation associated with an action $\mathrm{SL}(n, \mathbb{Z}) \curvearrowright X$ is $\{1\}$. Gaboriau proved in [Ga01] that the fundamental group of the orbit equivalence relation associated with an action $\mathbb{F}_n \curvearrowright X$ was also trivial for $2 \leq n < \infty$.

It follows immediately from the definitions that the fundamental group of the orbit equivalence relation $\mathcal{R}(G \curvearrowright X)$ is a subgroup of $\mathcal{F}(\mathrm{L}^\infty(X) \rtimes G)$. In many cases, one calculates the fundamental group of a group measure space II_1 factor by proving that equality holds. This is often done by proving uniqueness of a Cartan subalgebra, in which case isomorphism of the group measure space construction implies isomorphism of the orbit equivalence relation. Such uniqueness of Cartan results have been appearing more and more in the past few years, e.g. [OP07], [OP08], [CS11], [CSU11], [PV11], [PV12], [Io12], [DI12] and [Va13]. However, in [Po06, §6.1] Popa gives an example of a free ergodic probability measure preserving action of a countable group such that $\mathcal{F}(\mathcal{R}(G \curvearrowright X)) = \{1\}$ and $\mathcal{F}(\mathrm{L}^\infty(X) \rtimes G) = \mathbb{R}^*_+$, so the biggest possible difference can be realized. Earlier, Popa found free ergodic probability measure preserving actions for which $\mathcal{F}(\mathcal{R}(G \curvearrowright X))$ is countable, while the II_1 factor $\mathrm{L}^\infty(X) \rtimes G$ has fundamental group $\mathbb{R}^*_+$ ([Po90, Corollary of Theorem 3] and [Po86, Corollary 4.7.2]).

We would like to remark that there exist more invariants, both for II_1 factors and II_1 equivalence relations. For II_1 factors, one also studies (amongst others) the outer automorphism group and the category $\mathrm{Bimod}(M)$ of all finite index M-M-bimodules. For II_1 equivalence relation, other invariants include the cost of an equivalence relation and its L^2 Betti numbers. As we did not work with these invariants, we will not say more about them.

Description of the chapters and statement of the main results

In this thesis, we will look for examples of II_1 factors and underlying equivalence relations where there is a difference in fundamental group. We construct a group measure space II_1 factor $M = \mathrm{L}^\infty(X) \rtimes G$ with trivial fundamental group, but admitting a Cartan subalgebra A that is non-conjugate to $\mathrm{L}^\infty(X)$ and for which the associated equivalence relation has non-trivial fundamental group. This is not absurd, as the second Cartan inclusion is twisted by a 2-cocycle. To do this, we first need examples of II_1 factors with multiple Cartan subalgebras. The first example of a II_1 factor with two Cartan subalgebras that are not conjugate by an automorphism was given by Connes and Jones in [CJ82]. In [OP08, §7], Ozawa and Popa gave more explicit examples of this phenomenon. In fact, a II_1 factor can have uncountably many non-conjugate Cartan subalgebras (see [Po86], [Po90], [Po06] and [SV11]). By the previous paragraphs, the equivalence relation associated to the second Cartan inclusion $A \subset M$ must come with a non-trivial 2-cocycle. Indeed, otherwise its fundamental group would be a subset of $\mathcal{F}(M)$. Therefore our results are closely related to earlier constructions of II_1 factors with several Cartan subalgebras. Furthermore we give examples of McDuff II_1 factors coming from a group action, such that the associated equivalence relation can have as its fundamental group any group in a large class of groups. As they are McDuff II_1 factors, their fundamental group is $\mathbb{R}^*_+$.

In the first chapter we give a general introduction to group actions, countable equivalence relations and von Neumann algebras. We mention some interesting results in the field and look at special properties of groups and group actions that make the resulting equivalence relations and their fundamental groups easier to understand.

As we said, we want to construct a group measure space II_1 factor $M =$

$\mathrm{L}^\infty(X) \rtimes G$ with trivial fundamental group, but admitting a Cartan subalgebra A that is non-conjugate to $\mathrm{L}^\infty(X)$ and for which the associated equivalence relation has non-trivial fundamental group. So we first need examples of II_1 factors with multiple Cartan subalgebras. Secondly we need to be able to describe the equivalence relations associated to the second Cartan algebra. In the second chapter, we give a general setting that can provide examples of such situations in proving the following theorem.

Theorem A. *Let $\mathcal{Z}$ be a compact abelian group and $Z < \mathcal{Z}$ a countable subgroup. Let $Z_0 < Z$ be an infinite subgroup. Assume that Z acts on $\mathcal{Z}$ by translation. Let Γ be a countable group that acts on $\mathcal{Z}$ by continuous group automorphisms $(\alpha_g)_{g\in\Gamma}$ preserving Z and Z_0. Define*

$$M := \mathrm{L}^\infty(\mathcal{Z}) \rtimes (Z \rtimes \Gamma)\ .$$

Denote $\mathcal{Z}_0 := \overline{Z_0}$. Assume that for all $g \in \Gamma : \{z - \alpha_g(z) \mid z \in Z_0\}$ is either infinite or trivial. Denote $\Gamma_0 := \{g \in \Gamma \mid \alpha_g(z) = z \text{ for all } z \in Z_0\}$. Assume that $\{x \in \frac{\mathcal{Z}}{\mathcal{Z}_0} \mid \alpha_g(x) = x\}$ has infinite index in $\frac{\mathcal{Z}}{\mathcal{Z}_0}$ for all $g \in \Gamma_0 \backslash \{e\}$.

Assume finally that $Z \cap \mathcal{Z}_0 = Z_0$. Then $A := \mathcal{L}(Z_0) \mathbin{\overline{\otimes}} \mathrm{L}^\infty(\frac{\mathcal{Z}}{\mathcal{Z}_0}) = \mathrm{L}^\infty(\widehat{Z_0} \times \frac{\mathcal{Z}}{\mathcal{Z}_0})$ is a Cartan subalgebra of M and the induced equivalence relation on $\widehat{Z_0} \times \frac{\mathcal{Z}}{\mathcal{Z}_0}$ is given by the action $(\widehat{\mathcal{Z}_0} \times \frac{Z}{Z_0}) \rtimes \Gamma \curvearrowright \widehat{Z_0} \times \frac{\mathcal{Z}}{\mathcal{Z}_0}$, where $\widehat{\mathcal{Z}_0} \times \frac{Z}{Z_0}$ acts on $\widehat{Z_0} \times \frac{\mathcal{Z}}{\mathcal{Z}_0}$ by translation and Γ acts on both components in the natural way.

We will give some examples of group actions satisfying the conditions of the above theorem, and show that unfortunately there is no reason as to why there would be a difference in fundamental group. Finally, we give a setting that will prove to be more interesting. The concrete II_1 factor M that we construct will be an amalgamated free product (AFP). Several rigidity theorems, including computations of fundamental groups, for such AFP II_1 factors were obtained in [IPP05]. The setting of our example is the following.

Let $n \geq 6$ even and define Σ as

$$\Sigma := \begin{pmatrix} \mathrm{SL}_2(\mathbb{Z}) & 0 & \dots & 0 \\ 0 & \mathrm{SL}_2(\mathbb{Z}) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathrm{SL}_2(\mathbb{Z}) \end{pmatrix} < \mathrm{SL}_n(\mathbb{Z})\ .$$

Set

$$\Gamma = \mathrm{SL}_n(\mathbb{Z}) *_\Sigma (\Sigma \times \Lambda)\ ,$$

where Λ is a countable infinite group. Let $q : \Gamma \to \mathrm{SL}_n(\mathbb{Z})$ be the quotient map, i.e. $q(g) = g$ for all $g \in \mathrm{SL}_n(\mathbb{Z})$ and $q(\lambda) = e$ for all $\lambda \in \Lambda$. Fix a prime p and consider the action $\Gamma \overset{\alpha}{\curvearrowright} \mathbb{Z}_p^n$ through q. Remark that this action preserves $\mathbb{Z}^n$. Write $G := \mathbb{Z}^n \rtimes \Gamma$.

Define $X := (\mathbb{Z}_p^n)^\Gamma$. Embed $\mathbb{Z}_p^n$ into X by $i : \mathbb{Z}_p^n \to X : z \mapsto (\alpha_g(z))_{g\in\Gamma}$. Now consider the action $G \curvearrowright X$, where $\mathbb{Z}^n$ acts by translation after embedding by i, and Γ acts by Bernoulli shift. Denote by M

$$M := \mathrm{L}^\infty(X) \rtimes G = \mathrm{L}^\infty((\mathbb{Z}_p^n)^\Gamma) \rtimes (\mathbb{Z}^n \rtimes (\mathrm{SL}_n(\mathbb{Z}) *_\Sigma (\Sigma \times \Lambda)))$$

the associated group measure space II_1 factor.

In chapters three and four, we prove the following theorem

Theorem B. *With $G \curvearrowright X$ as above, $M = \mathrm{L}^\infty(X) \rtimes G$ has at least two Cartan subalgebras $A_1 = \mathrm{L}^\infty(X)$ and $A_2 = \mathrm{L}^\infty\left(\frac{(\mathbb{Z}_p^n)^\Gamma}{i(\mathbb{Z}_p^n)}\right)\overline{\otimes}\mathcal{L}(\mathbb{Z}^n)$that are not conjugate by an automorphism of M. Denote the associated equivalence relations by $\mathcal{R}_1$ and $\mathcal{R}_2$. Then the following holds:*

1. $\mathcal{F}(M) = \mathcal{F}(\mathcal{R}_1) = \{1\}$,
2. $\mathcal{F}(\mathcal{R}_2) = \{p^{kn} \mid k \in \mathbb{Z}\}$.

The first part will follow from Theorem 3.1, which we will prove in chapter three. This will be done by exploiting the amalgamated free product structure of the example: our proof that $\mathcal{F}(M) = \{1\}$ uses the techniques of [IPP05] and the recent work of [Io12] on Cartan subalgebras in AFP II_1 factors. This will allow us to describe partial automorphisms of M. Then we will use cocycle superrigidity techniques (§3.2) to conclude that theorem 3.1 holds. One key ingredient will be showing that on a part of M, a stable automorphism can be seen as a stable orbit equivalence of a group action. Another crucial part will be a twisted version of Popa's cocycle superrigidity theorem [Po05, Theorem 5.5] for Bernoulli actions of groups admitting an infinite rigid subgroup that is wq-normal.

The assertion that M has a second Cartan subalgebra, as well as the second part of Theorem B,will be proven in chapter four. The first part will amount to showing that the example satisfies the setting of chapter 2. The second part of Theorem B will be proven using [PV08b, Lemma 5.10], after showing that a 'lift' of the group action defining the second equivalence relation is $\mathcal{U}_{fin}$-cocycle superrigid. To prove this cocycle superrigidity, we use [PV08b, Theorem 5.3].

Finally, in chapter five, we extend Popa's example from [Po06, §6.1]. There, Popa gives a group action $\Gamma \curvearrowright X$ such that the fundamental group of the equivalence relation is $\{1\}$ whereas the associated II_1 factor is McDuff. We first show that under certain conditions the fundamental group of the orbit equivalence relation coming from a direct product of two actions is ‘the product of the fundamental groups of the orbit equivalence relations coming from each of the actions’ (Theorem 5.4). We then apply this theorem to Popa’s example and the results from [PV08a]. There Popa and Vaes give examples of group actions such that the resulting equivalence relation and II_1 factor can have ‘arbitrary’ fundamental group, where by arbitrary we mean any group on a large class of groups $\mathcal{S}_{\mathrm{centr}}$ (see [PV08a, §2]). This large class of groups contains groups of the form $\exp(H)$ where H can be an uncountable group with any Hausdorff dimension $\alpha \in (0, 1)$. This leads to the following theorem.

Theorem C. *Let* $\Gamma \curvearrowright Y$ *be the group action defined in Popa’s example [Po06, §6.1], and* $\Gamma_0 < \Gamma$ *the infinite property (T) subgroup. For any* $F \in \mathcal{S}_{\mathrm{centr}}$ *(see [PV08a, §2]), there are actions* $\mathbb{F}_\infty \curvearrowright X$ *such that* $\mathcal{F}(\mathcal{R}((\mathbb{F}_\infty \times \Gamma) \curvearrowright (X \times Y))) = F$ *and* $\mathcal{F}(\mathrm{L}^\infty(X \times Y) \rtimes (\mathbb{F}_\infty \times \Gamma)) = \mathbb{R}^*_+$.

Chapter 1

Preliminaries

1.1 Group actions

1.1.1 Basic facts

Most examples of II_1 factors given throughout this thesis are constructed from group actions. When we speak of group actions, it is important to know on what kind of spaces our groups will act, and in what kind of way. The spaces will be 'nice' in the following sense.

Definition 1.1.
- A Polish space is a separable, complete, metrizable space.
- A standard Borel space is a measurable space $(X, \mathcal{B})$ that is isomorphic to a Borel subset of a Polish space.
- A standard measure space (X, μ) is a standard Borel space with a measure on the Borel sets. If the measure has mass 1, (X, μ) is called a standard probability space.

Examples of Polish spaces include $\mathbb{R}^n$ and $\mathbb{C}^n$ with the usual topology, and many more. However in a measurable sense many standard probability spaces are the same, as is illustrated by the following theorem.

Theorem 1.2. *[Ke95, Theorem 17.41] (X, μ) is a standard non-atomic probability space if and only if there exists a Borel isomorphism $f : X \to [0, 1]$ such that $f\mu = \lambda_{|[0,1]}$.*

Throughout we will work with actions of locally compact second countable (l.c.s.c.) groups on standard measure spaces (and often even actions of countable discrete groups on standard probability spaces). For this, we need to properly define what we mean by an automorphism of a standard measure space in this context.

Definition 1.3. Let (X, μ) be a standard measure space. An automorphism of (X, μ) is a bi-measurable Borel bijection preserving null sets, defined up to equality almost everywhere. We denote by $\mathrm{Aut}(X)$ the group of all automorphisms of (X, μ).

Note that $\mathrm{Aut}(X)$ has a natural topology. We will say that a net $(\alpha_i)_i$ of automorphisms of (X, μ) converges to α whenever for all measurable non-negligible subset $U \subset X$ we have $\mu(\alpha_i(U) \,\Delta\, \alpha(U)) \to 0$ (where Δ denotes the symmetric difference). Endowed with this topology, $\mathrm{Aut}(X)$ is a Polish space.

Definition 1.4. Let G be a l.c.s.c. group and (X, μ) a standard measure space. A group action $\alpha : G \curvearrowright (X, \mu)$ is a continuous group morphism $\alpha : G \to \mathrm{Aut}(X)$ (where $\mathrm{Aut}(X)$ is endowed with the above mentioned topology).

An example of such a group action on a standard probability space is the action of $\mathrm{SL}_2(\mathbb{Z})$ on the 2-torus $\mathbb{T}^2$ by setting $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^a y^b \\ x^c y^d \end{pmatrix}$.

Working with actions on measure spaces, we want the action to preserve the measure. We say an action $G \curvearrowright (X, \mu)$ is non-singular if for all $A \subset X$ one has $\mu(g \cdot A) = 0 \Leftrightarrow \mu(A) = 0$. The action is called measure preserving if $\mu(g \cdot A) = \mu(A)$ for every measurable subset $A \subset X$. From now on we will assume all actions to be non-singular, unless mentioned otherwise. We will also be interested in the following special kinds of actions.

Definition 1.5. Let $G \curvearrowright (X, \mu)$ be an action of a l.c.s.c. group on a standard measure space.

- For any $x \in X$ define $\mathrm{Stab}(x) = \{g \in G \mid g \cdot x = x\}$. One can show (see e.g. [MRV11, Lemma 10] for a proof) that the set $X_0 := \{x \in X \mid \mathrm{Stab}(x) = \{e\}\}$ is a G-invariant Borel subset of X. We say the action is essentially free if $\mu(X_0^c) = 0$.
- We say the action is ergodic if every measurable subset $A \subset X$ that is invariant under the action satisfies $\mu(A)\mu(A^c) = 0$.

Alternatively, an action is ergodic if for every measurable set $A \subset X$ its saturation $\mathcal{S}(A) = \cup_{g \in G} g \cdot A$ (or a measurable conegligible subset) is conegligible.

Existence of such a measurable conegligible subset is shown in [Zi84, Lemma B.8]. Note that essential freeness means that there are (in a measurable setting) no fixed points. Ergodicity means that the action does not 'split'.

Example 1.6.
- Let $\Gamma < G$ be a countable subgroup of a compact group with Haar measure μ. Suppose Γ is dense in G. Then one can show that the action $\Gamma \curvearrowright G$ by translation is ergodic and measure preserving.
- The action $\mathbb{Z} \curvearrowright \frac{\mathbb{R}}{r\mathbb{Z}}$ by translation, with $r \notin \mathbb{Q}$, is free and ergodic.

Both in chapter 3 and chapter 4 we will need the notion of induced actions. We recall the definition and show that if a countable discrete group Γ acts on a standard measure space X such that the diagonal action $\Gamma \curvearrowright X \times X$ is ergodic, then the action is not induced.

Definition 1.7. Let $\Gamma \overset{\alpha}{\curvearrowright} (X, \mu)$ be an essentially free, ergodic, non-singular action of a countable group on a standard measure space. We say that α is induced from $\Gamma_0 \curvearrowright X_0$, if Γ_0 is a subgroup of Γ, X_0 is a non-negligible subset of X and $g \cdot X_0 \cap X_0$ is negligible for all $g \in \Gamma - \Gamma_0$.

Lemma 1.8. *Let $\Gamma \overset{\alpha}{\curvearrowright} (X, \mu)$ be an essentially free, ergodic, non-singular action of a countable group on a standard measure space and assume that $\Gamma \curvearrowright X \times X$ is ergodic. If α is induced from $\Gamma_0 \curvearrowright X_0$, then $\Gamma = \Gamma_0$ and $X \backslash X_0$ is negligible.*

Proof. Assume α is induced from $\Gamma_0 \curvearrowright X_0$. Let $\pi : X \to \Gamma/\Gamma_0$ be the quotient map such that $X_0 = \pi^{-1}(e\Gamma_0)$ and $\pi(g \cdot x) = g\pi(x)$. Set $A := \{(x, y) \in X \times X \mid \pi(x) = \pi(y)\}$. By construction A is non-negligible (as $X_0 \times X_0 \subset A$) and A is Γ-invariant. By ergodicity of $\Gamma \curvearrowright X \times X$, $\mu(X \times X - A) = 0$ and hence $\pi(x) = \pi(y)$ for almost all $(x, y) \in X \times X$. In particular $\mu(X - X_0) = 0$, so $\Gamma = \Gamma_0$. $\square$

Finally we would like to remark that whenever we have an action $G \curvearrowright (X, \mu)$ of a l.c.s.c. group G on a standard probability space X, this immediately gives an action of G on $\mathrm{L}^\infty(X, \mu)$. This follows from the fact that $\mathrm{Aut}(X, \mu)$ is $\mathrm{Aut}(\mathrm{L}^\infty(X, \mu))$ in a canonical way: whenever $\Delta \in \mathrm{Aut}(X, \mu)$, set $\Phi_\Delta : \mathrm{L}^\infty(X) \to \mathrm{L}^\infty(X) : F \mapsto F \circ \Delta$. The converse is also possible, i.e. whenever we have an action of a locally compact second countable group G on $\mathrm{L}^\infty(X, \mu)$, this translates to an action of G on (X, μ) (see [Zi84, Theorem B.10]). With this 'new' action, one can show that ergodicity of $G \curvearrowright (X, \mu)$ is equivalent to G-invariant functions $F \in \mathrm{L}^\infty(X)$ being constant almost everywhere.

1.1.2 Different types of mixing

Above we already gave a few basic properties of group actions. Most of the actions we will consider will satisfy these. Another, stronger property of group actions that will be very useful, is the mixing property. This property is a strengthening of the notion of ergodicity, and comes in different strengths (which are appropriately called weakly, mildly and strongly mixing).

Definition 1.9. [Sch84, §2] Let G be a countable group, (X, μ) a standard probability space and $G \curvearrowright X$ a probability measure preserving, ergodic action. We say that $G \curvearrowright X$ is

1. *weakly mixing* if for every $\epsilon > 0$, every $n \in \mathbb{N}$ and all measurable $A_1, \ldots, A_n \subset X$ there is a $g \in G$ such that for all $i, j \in \{1, \ldots, n\}$ we have
$$|\mu(A_i \cap gA_j) - \mu(A_i)\mu(A_j)| < \epsilon \ .$$
2. *mildly mixing* if for every measurable set $B \subset X$ with $0 < \mu(B) < 1$,
$$\liminf_{g \to \infty} \mu(B \Delta gB) > 0 \ ,$$
3. *strongly mixing* if for all measurable sets $A, B \subset X$ we have
$$\lim_{g \to \infty} \mu(A \cap gB) = \mu(A)\mu(B) \ .$$

One can see that, whereas ergodicity of $G \curvearrowright (X, \mu)$ says that 'every' subset of X passes everywhere in X under the action, the mixing properties say something about the fact that this is done in a more or less 'uniform' way. In particular, all of these properties immediately imply ergodicity.

There are many equivalent definitions known for these properties. One is in terms of the Koopman representation.

Definition 1.10. Let $G \curvearrowright (X, \mu)$ be a probability measure preserving action on a standard probability space. The Koopman representation of G associated to this action is the unitary representation $\pi : G \to \mathcal{U}(\mathrm{L}^2_0(X, \mu))$ given by $(\pi(\gamma)f)(x) = f(\gamma^{-1}x)$, where $\mathrm{L}^2_0(X, \mu) = \{f \in \mathrm{L}^2(X, \mu) \mid \int_X f d\mu = 0\}$.

One can define mixing for unitary representations of groups.

Definition 1.11. Let G be a countable group, $\pi : G \to \mathcal{U}(\mathcal{H})$ a unitary representation of G on a Hilbert space $\mathcal{H}$. We say that π is

1. weakly mixing if for every finite $F \subset \mathcal{H}$ and every $\epsilon > 0$ there is $g \in G$ such that $|\langle \pi(g)\xi, \xi \rangle| < \epsilon$, for all $\xi \in F$.

2. mildly mixing if there is no non-zero Hilbert subspace $\mathcal{K} \subset \mathcal{H}$ such that there is a sequence $(g_n)_n$ of elements in G tending to infinity and satisfying $\pi(g_n)\xi \to \xi$ for all $\xi \in \mathcal{K}$.

3. strongly mixing if for each finite set $F \subset \mathcal{H}$ we have $\lim_{g \to \infty} |\langle \pi(g)\xi, \xi \rangle| = 0$ for all $\xi \in F$.

One can show that for the Koopman representation each of these is equivalent with the action being weakly, mildly or strongly mixing (see e.g. [Ra12, Proposition 2.1] for mildly mixing, and [Pe11, Proposition 2.2.12] resp. [Pe11, Proposition 2.2.15] for weakly resp. strongly mixing).

Example 1.12.
- The action of $\mathrm{SL}_2(\mathbb{Z})$ on the torus $\mathbb{T}^2$ is weakly mixing. Indeed, as $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{Z}^2 - \{(0,0)\}$ has infinite orbits, the Koopman representation is weakly mixing, and so is the action on the torus. Note that this action is not mildly mixing.

- Recall that a matrix is called unipotent if all its eigenvalues are equal to 1. The action of a subgroup H of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{T}^2$ is strongly mixing if and only if H contains no non-trivial unipotent elements ([BG04, Proposition 2.30]).

In the next subsection we will give a class of examples that all satisfy the strong mixing property. Using the following proposition, one can see that the choice of names is indeed appropriate.

Proposition 1.13. *[Sch84, Proposition 2.3] Let $G \curvearrowright (X, \mu)$ be a probability measure preserving, ergodic action of a countable group on a standard probability space. Then $G \curvearrowright (X, \mu)$ is mildly mixing if and only if for every non-singular, properly ergodic (i.e. every G-orbit has measure zero) action $G \curvearrowright (Y, \nu)$ on a σ-finite standard measure space, the product action $G \curvearrowright (X \times Y, \mu \times \nu)$ is ergodic.*

If we restrict $G \curvearrowright (Y, \nu)$ to probability measure preserving actions on standard probability spaces, Proposition 1.13 gives a characterization of weakly mixing actions. It is now clear that $3 \Rightarrow 2 \Rightarrow 1$ in Definition 1.9. Reverse arrows do not hold, as is shown in [Sch84, Theorem 4.2] and [Sch84, Theorem 4.4]. There Schmidt gives examples of Gaussian actions of l.c.s.c. non-compact abelian groups that are weakly but not mildly, resp. mildly but not strongly mixing (which is more subtle).

1.1.3 Bernoulli actions

A very interesting (and rather well understood) example of an action on a standard probability space is the so called generalized Bernoulli action. Whenever we have a countable group Γ acting on a countable set K and a 'base space' (X_0, μ_0) which is a standard probability space, we can build an action $\Gamma \curvearrowright (X_0, \mu_0)^K$ by shifting the index, i.e. $g \cdot (x_i)_{i \in K} = (x_{g^{-1} \cdot i})_i \in K$. We will always assume that for $g \neq e$ the set $\{i \mid g \cdot i \neq i\}$ is infinite (this assures that the action is free). In the case that $I = \Gamma$ and $\Gamma \curvearrowright \Gamma$ by translation, we get the Bernoulli action.

This generalized Bernoulli action is very interesting, as it has many of the above mentioned properties. It is clearly measure preserving. One can show that the action is essentially free (though it is not free, but the set of fixed points has measure zero) and ergodic (this can be shown to be equivalent with requiring that $\Gamma \cdot i$ is infinite for all $i \in K$). In fact, requiring that $\Gamma \cdot i$ is infinite for all $i \in K$ assures the action is even weakly mixing. Furthermore for a generalized Bernoulli action, being strongly mixing is equivalent to being mildly mixing. This is the case if and only if stabilizers of finite subsets $K_0 \subset K$ are finite. Because of these nice properties, Bernoulli actions are 'nice' to work with.

1.2 Countable equivalence relations

In this section we will formally define what is meant by a countable (Borel) equivalence relation on a standard Borel space.

Definition 1.14. Let $(X, \mathcal{B})$ be a standard Borel space. An equivalence relation $\mathcal{R} \subset X \times X$ is called a countable Borel equivalence relation if

- the equivalence classes are countable
- $\mathcal{R} \subset X \times X$ is a Borel subset for the product σ-algebra.

We will always assume the measure to be non-singular. Whenever we have an equivalence relation $\mathcal{R}$ on a standard measure space (X, μ) and a measurable subset $U \subset X$, one can define the restricted equivalence relation $\mathcal{R}_{|U}$ as $\{(x, y) \mid x, y \in U, x \sim_{\mathcal{R}} y\}$.

Many group properties can be translated to a context of countable equivalence relations. To define ergodicity in this context we need to define the saturation of a Borel set first.

Definition 1.15. Let $B \subset X$ be a Borel subset. The saturation of B is defined as $\mathcal{S}(B) = \cup_{x \in B}\{y \mid (x,y) \in \mathcal{R}\}$.

One can show that this is indeed a Borel subset of X. This allows us to define ergodicity.

Definition 1.16. Let $\mathcal{R}$ be a countable Borel equivalence relation on a standard measure space (X,μ) (called a standard equivalence relation from here on out). We say that $\mathcal{R}$ is ergodic if for every measurable subset $B \subset X$ either $\mu(\mathcal{S}(B)) = 0$ or $\mu(\mathcal{S}(B)^c) = 0$.

Note that this is equivalent to saying that whenever $F \in \mathrm{L}^\infty(X)$ is invariant under $\mathcal{R}$ (i.e. $F(x) = F(y)$ whenever $x \sim_{\mathcal{R}} y$) implies F is constant almost everywhere (as in the group action case).

To define what is meant by a measure preserving equivalence relation, we will define two measures on the equivalence relation itself.

Definition 1.17. Let $\mathcal{R} \subset X \times X$ be a standard equivalence relation where $X = (X,\mu)$ is a standard measure space. We define the left and right counting measures on Borel subsets of $\mathcal{R}$ as follows:

$$\begin{aligned}
\mu_l(A) &= \int_X \#\{y \in X \mid (x,y) \in A\} d\mu(x) \\
\mu_r(A) &= \int_X \#\{x \in X \mid (x,y) \in A\} d\mu(y)
\end{aligned}$$

A countable equivalence relation on a standard measure space is said to preserve the measure μ if $\mu_l = \mu_r$. We will denote this measure on $\mathcal{R}$ by $\mu^{(1)}$.

The following lemma shows that this is indeed a good definition of measure preserving. A partial Borel isomorphism is a Borel isomorphism between non-negligible measurable subsets A, B of X.

Lemma 1.18. *Let $\mathcal{R}$ be a countable equivalence relation on (X,μ). $\mathcal{R}$ preserves μ if and only if for every partial Borel isomorphism $\varphi : A \to B$ with $\mathrm{Gr}(\varphi) \subset \mathcal{R}$ we have $\mu_{|\mathrm{rng}\varphi} \circ \varphi = \mu_{|\mathrm{dom}\varphi}$.*

For every n one can extend a measure preserving equivalence relation on a standard measure space (X,μ) in the following way. Let $\mathcal{R}^{(n)} = \{(x_1,\dots,x_{n+1}) \mid (x_1,x_2) \in \mathcal{R},\dots,(x_n,x_{n+1}) \in \mathcal{R}\}$. This space is endowed with an invariant measure $\mu^{(n)}$ defined as

$$\mu_i^{(n)}(A) = \int_X \#\{(x_1,\dots,x_{i-1},x_{i+1},\dots,x_{n+1}) \mid (x_1,x_2,\dots,x_{n+1}) \in A\} d\mu(x_i).$$

As the equivalence relation is measure preserving, this does not depend on the choice of the variable x_i.

These notions of ergodicity and preservation of measure allow us to classify countable equivalence relations in types:

Definition 1.19. Let $\mathcal{R}$ be an ergodic non-singular countable equivalence relation on a standard measure space (X, μ).

- If μ is an atomic measure, $\mathcal{R}$ is said to be of type I.
- If there is a finite non-atomic $\mathcal{R}$-invariant measure ν that is equivalent to μ, $\mathcal{R}$ is of type II_1.
- If there is a σ-finite infinite non-atomic $\mathcal{R}$-invariant measure ν that is equivalent to μ, $\mathcal{R}$ is of type II_∞.
- If there is no such σ-finite $\mathcal{R}$-invariant measure equivalent to μ, $\mathcal{R}$ is of type III.

Finally we distinguish a 'special' property of equivalence relations: the so called hyperfiniteness. (This notion also exists for von Neumann algebras, we will come back to this in the next section).

Definition 1.20. Let $\mathcal{R}$ be a countable equivalence relation on a standard Borel space. We say that $\mathcal{R}$ is hyperfinite if it can be written as the union of a countable increasing sequence of standard finite equivalence relations.

Recall that a group is amenable if it admits a left-invariant mean. Examples of amenable groups include all solvable groups, and the class of amenable groups is closed under taking subgroups, quotients, direct limits and group extensions. The following theorem states that the equivalence relation of any action of an amenable group on a standard probability space is hyperfinite.

Theorem 1.21. *[OW80] Suppose G is a countable amenable group and X is a standard Borel space. If μ is any Borel-probability measure on X, there exists a Borel subset $Y \subset X$ with $\mu(Y) = 1$ and such that $\mathcal{R}(G \curvearrowright Y)$ is hyperfinite.*

As was said in the introduction, whenever we have a countable group acting on a standard measure space, this gives us a countable equivalence relation. In the next section we will look a little bit deeper into that. The following theorem shows us that the converse also holds.

Theorem 1.22. *[FM75] Let $\mathcal{R}$ be a μ-preserving countable standard equivalence relation on a non-atomic standard measure space (X, μ). Then there exists a countable group Γ such that $\mathcal{R} = \mathcal{R}_\Gamma$.*

However, there are situations where this group is not obvious to find. Furthermore it was not clear if these actions needed to be essentially free. It turned out not to be so, as was shown in the following theorem by Furman.

Theorem 1.23. *[Fu99, Theorem D] Let $\mathcal{R}$ be the orbit equivalence relation generated by the natural action $\mathrm{SL}_n(\mathbb{Z}) \curvearrowright \mathbb{T}^n$ for $n \geq 3$. Let $A \subset \mathbb{T}^n$ be a measurable subset such that $\mu(A) \notin \mathbb{Q}$. Then $\mathcal{R}_{|A}$ can not be generated by a free action of a countable group.*

1.2.1 Orbit equivalence relations

In this subsection, we will focus on countable equivalence relations arising from actions of countable groups on standard probability spaces. We will focus on actions that are essentially free, ergodic and p.m.p. (so the associated equivalence relation is of type II_1.) We define the orbit equivalence relation for any action though.

Definition 1.24. Let $G \curvearrowright (X, \mu)$ be an action of a countable group on a standard measure space. The orbit equivalence relation (denoted $\mathcal{R}(G \curvearrowright X)$) is defined as $\mathcal{R}(G \curvearrowright X) := \{(x, y) \in X \times X \mid \exists g \in G : g \cdot x = y\}$.

One checks that ergodic measure preserving actions give rise to ergodic measure preserving equivalence relations. A natural question to be asked, is whether given an equivalence relation coming from such a group action, one is able to reconstruct the group. Before we look into this, let us give different ways of equivalence relations to be ‘equivalent’.

Definition 1.25. Let $\Gamma \overset{\alpha}{\curvearrowright} (X, \mu)$ and $\Lambda \overset{\beta}{\curvearrowright} (Y, \nu)$ be essentially free, ergodic, non-singular actions of countable groups on standard measure spaces.

- Let $\delta : \Gamma \to \Lambda$ be a group isomorphism. A δ-conjugacy of the actions α and β is a non-singular isomorphism $\Delta : X \to Y$ such that $\Delta(g \cdot x) = \delta(g) \cdot \Delta(x)$ for all $g \in \Gamma$ and a.e. $x \in X$.
- A stable orbit equivalence (sOE) between the actions α and β is a non-singular isomorphism $\Delta : X_0 \to Y_0$ between non-negligible subsets $X_0 \subset X, Y_0 \subset Y$, such that Δ is an isomorphism between the restricted orbit equivalence relations $\mathcal{R}(\Gamma \curvearrowright X)_{|X_0}$ and $\mathcal{R}(\Lambda \curvearrowright Y)_{|Y_0}$. We say that α and β are stably orbit equivalent. If the actions are measure preserving, a stable orbit equivalence scales the measure: the compression constant of Δ is defined as $c(\Delta) := \frac{\nu(Y_0)}{\mu(X_0)}$. If $\mu(X_0^c) = \nu(Y_0^c) = 0$ we say that α and β are orbit equivalent (OE).

Whenever there is a stable orbit equivalence Δ for free ergodic probability measure preserving actions $G \curvearrowright X$ and $H \curvearrowright Y$, we can choose a map $\Theta : X \to X_0$ such that $\Theta(x) \in G \cdot x$ for almost all $x \in X$. Writing $\Delta_0 := \Delta \circ \Theta$, Δ_0 is a local isomorphism from X to Y, i.e. Δ_0 is a Borel map and X can be partitioned into a sequence of non-negligible subsets $X_n \subset X$ such that the restriction of Δ_0 to any of these subsets X_n is a non-singular measure space isomorphism of X_n onto some non-negligible subset of Y. By construction Δ_0 is orbit preserving, i.e. for almost all $x, y \in X$ we have $x \in G \cdot y$ if and only if $\Delta_0(x) \in H \cdot \Delta_0(y)$.

Whenever $G \curvearrowright X$ is induced from $G_0 \curvearrowright X_0$, one can show that $G \curvearrowright X$ is stably orbit equivalent to $G_0 \curvearrowright X_0$, and the compression constant is $[G : G_0]^{-1}$.

It is clear that whenever two actions are conjugate, they are also orbit equivalent. The reverse, however, need not be true in general. Results that prove that the reverse implication does hold for certain (classes of) groups or actions, are called 'orbit equivalence rigidity' results. It follows that if this reverse implication does hold, we can indeed reconstruct the original group action starting from the orbit equivalence relation. However in some cases almost all information about the underlying group is lost. For example, all orbit equivalence relations coming from actions of amenable groups on standard probability spaces are orbit equivalent: Dye showed in [Dy59,] that all ergodic probability measure preserving actions of $\mathbb{Z}$ are mutually orbit equivalent. He also observed that all ergodic hyperfinite probability measure presrving actions are mutually orbit equivalent. So from Theorem 1.21 it follows that indeed all orbit equivalence relations coming from actions of amenable groups on standard probability spaces are orbit equivalent. To give a flavor of orbit equivalence superrigidity results, we will mention two results in this theory. First, we need the notion of virtually isomorphic actions.

Definition 1.26. [Fu99, Definition 1.1] Two countable groups Γ_1 and Γ_2 are virtually isomorphic groups if there exist finite normal subgroups $N_i \lhd \Gamma_i$ such that the quotient groups $\Gamma_i' = \Gamma_i/N_i$ contain isomorphic subgroups of finite index: $\Gamma_1'' \cong \Gamma_2''$ where $[\Gamma_i' : \Gamma_i''] < \infty$. The ergodic probability measure preserving actions $\Gamma_1 \curvearrowright X_1$ and $\Gamma_2 \curvearrowright X_2$ are virtually conjugate actions if Γ_1 and Γ_2 are virtually isomorphic groups, and for some choice of N_i, Γ_i'' the actions $\Gamma_1'' \curvearrowright X_1''$ and $\Gamma_2'' \curvearrowright X_2''$ are conjugate, where X_i'' is one of the at most $[\Gamma_i' : \Gamma_i'']$-many mutually isomorphic Γ_i''-ergodic components.

Theorem 1.27. *[Fu99, Theorem A] Any free, ergodic, probability measure preserving action that is OE with the standard action* $\mathrm{SL}_n(\mathbb{Z}) \curvearrowright \mathbb{T}^n$ *for* $n \geq 3$ *is virtually conjugate with it.*

Theorem 1.28. *[Po05, Theorem 5.6] Assume* Γ *is either an infinite ICC property (T) group (see* §1.5*) or the product of two nonamenable infinite groups*

$H \times H'$ and has no finite normal subgroup. Then any free action that is orbit equivalent with the Bernoulli shift $\Gamma \curvearrowright (X_0, \mu_0)^\Gamma$ is conjugate with it.

In the next section, we briefly come back to this notion of superrigidity, but in the context of von Neumann algebras.

1.3 von Neumann algebras

In this section we give a brief introduction to von Neumann algebras. Analytically speaking, von Neumann algebras are unital $*$-subalgebras of $\mathrm{B}(\mathcal{H})$ that are closed for the weak operator topology. By von Neumann's bicommutant theorem, von Neumann algebras can also be characterized in a purely algebraic way. To do this, we first define, for a subset $M \subset \mathrm{B}(\mathcal{H})$, its commutant M' as $\{x \in \mathrm{B}(\mathcal{H}) \mid xy = yx \ \ \forall y \in M\}$.

Theorem 1.29. *[vN29] Let $M \subset \mathrm{B}(\mathcal{H})$ be a unital $*$-subalgebra. Then M is a von Neumann algebra if and only if $M = (M')'$.*

Sakai showed in [Sa71] that von Neumann algebras can also be defined abstractly as C^*-algebras that have a predual, i.e. von Neumann algebras can be seen as the dual Banach space of some other Banach space, which is called the predual. From now on, we will always assume this predual to be separable. One can show that a von Neumann algebra with separable predual is isomorphic to a subalgebra of $\mathrm{B}(\mathcal{H})$ for some separable Hilbert space $\mathcal{H}$.

As was said before, Murray and von Neumann proved in [vN49] that to classify von Neumann algebras it suffices to classify von Neumann algebras with trivial center, i.e. those whose center only contains scalar multiples of the identity. These von Neumann algebras are called factors. In fact they showed that every von Neumann algebra can be decomposed as a direct integral of factors. These factors are classified in types I, II and III. Before we give this classification, we need two important definitions.

Definition 1.30. A positive linear functional $\tau : M \to \mathbb{C}$ on a von Neumann algebra M is called a state if $\tau(1) = 1$. A state is called tracial if $\tau(xy) = \tau(yx)$ for all $x, y \in M$. It is faithful if $\tau(x^*x) = 0$ implies $x = 0$ for every $x \in M$. Finally, it is normal if it is continuous for the ultraweak topology.

Note that on a II_1 factor, faithfulness of normal traces is automatic. Whenever (M, τ) is a von Neumann algebra with a faithful normal finite trace τ, one can show that there exists a (up to unitary equivalence) unique representation λ_τ :

$M \to \mathrm{B}(\mathcal{H}_\tau)$ on the Hilbert space $\mathcal{H}_\tau$ with a vector ξ_τ satisfying $\overline{\lambda_\tau(M)\xi_\tau} = \mathcal{H}_\tau$ and $\tau(a) = \langle \xi_\tau, \lambda_\tau(a)\xi_\tau \rangle$. This is the GNS construction for von Neumann algebras. We will denote this Hilbert space by $\mathrm{L}^2(M)$, and denote by $\|\cdot\|_2$ the norm inherited from the inner product. We will also need the notion of infinite traces.

Definition 1.31. A map $\mathrm{Tr} : M^+ \to [0, +\infty]$ is called a trace if

1. $\mathrm{Tr}(x + y) = \mathrm{Tr}(x) + \mathrm{Tr}(y)$ for all $x, y \in M^+$,
2. $\mathrm{Tr}(\lambda x) = \lambda \, \mathrm{Tr}(x)$ for all $x \in M^+$, $\lambda \in \mathbb{R}^+$,
3. $\mathrm{Tr}(x^* x) = \mathrm{Tr}(x x^*)$ for all $x \in M$.

It is called semifinite if for every non-zero $x \in M^+$ there is a non-zero $y \in M^+$ such that $y \leq x$ and $\mathrm{Tr}(y) < +\infty$. It is called normal if $\mathrm{Tr}(\sup_i x_i) = \sup_i \mathrm{Tr}(x_i)$ for every bounded increasing net (x_i) in M^+. Finally it is called faithful if $\mathrm{Tr}(x) = 0$ if and only if $x = 0$.

Now existence of these traces will allow us to classify von Neumann algebras into types.

Definition 1.32. Let M be a factor. We say that

- M is a type I factor if and only if $M \cong \mathrm{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.
- M is a type II_1 factor if and only if M is infinite dimensional and has a finite normal tracial state.
- M is a type II_∞ factor if and only if M has a semi-finite trace Tr, such that $M \ncong \mathrm{B}(\mathcal{H})$ for any infinite dimensional Hilbert space, and $\mathrm{Tr}(1) = +\infty$.
- M is a type III factor if every normal semifinite trace is zero.

This classification into types can be shown to be the same as the one we mentioned in the introduction. Note that whenever M is a II_1 factor the von Neumann algebra $\mathrm{M}_n(M)$, defined as $n \times n$ matrices with entries in M, is still a II_1 factor. This allows us to define, for any $t > 0$, an amplification of M as $M^t := p\mathrm{M}_n(M)p$ where $(\mathrm{Tr} \otimes \tau)(p) = t$. One can show that for any $s, t > 0$ one has $(M^t)^s = M^{ts}$.

Finally, before we proceed to giving examples of von Neumann algebras, we define the notion of a Cartan subalgebra. This notion will prove to be very important in studying the link between II_1 factors and II_1 equivalence relations.

Definition 1.33. Let (M, τ) be a tracial von Neumann algebra and $A \subset M$ be a *-subalgebra of M. We say that A is a Cartan subalgebra of M if A is maximal abelian in M (i.e. $A' \cap M = A$) and the normalizer $\mathcal{N}_M(A) := \{u \in \mathcal{U}(M) \mid uAu^* = A\}$ of A in M generates M as a von Neumann algebra.

1.3.1 Examples of von Neumann algebras

From the definition, it is immediately clear that $\mathrm{B}(\mathcal{H})$ is a factor, albeit not the most interesting one. Furthermore $\mathrm{L}^\infty(X, \mu)$ is also a von Neumann algebra, and it can be shown that any abelian von Neumann algebra with separable predual is of the form $\mathrm{L}^\infty(X, \mu)$ for some standard probability space (X, μ). However there are many more von Neumann algebras. We will briefly give two ways of 'building' von Neumann algebras from other mathematical objects: the group and equivalence relation von Neumann algebra, and the group measure space construction. As the names imply, the first is built from a group or an equivalence relation, and the second is built from a group acting on a measure space.

Before we do this, we look at the notion of hyperfiniteness in the von Neumann algebra setting. In [MvN43], Murray and von Neumann proved that there exists a unique hyperfinite II_1 factor (up to isomorphism). This hyperfinite II_1 factor is defined as the bicommutant of an increasing union of matrix algebras. We will denote this hyperfinite II_1 factor always by R. A deep result by Connes in [Co76] showed that hyperfiniteness is equivalent to the II_1 factor $M \subset \mathrm{B}(\mathcal{H})$ being injective, i.e. there exists a conditional expectation of $\mathrm{B}(\mathcal{H})$ onto M.

Closely related is a class of II_1 factors that are called McDuff II_1 factors. These factors will be very useful in chapter 5.

Definition 1.34. Let M be a II_1 factor.

- A bounded sequence $(x_k)_k \subset M$ is called a central sequence if for all $x \in M$ we have $\|x_k x - x x_k\|_2 \to 0$.
- A central sequence $(x_k)_k \subset M$ is called hypercentral if for all central sequences $(y_k)_k \in M$ we have $\|x_k y_k - y_k x_k\|_2 \to 0$.
- M is called a McDuff II_1 factor if not all central sequences are hypercentral.

These McDuff II_1 factors have a very special property, as is shown by the following theorem.

Theorem 1.35. *[McD69, Theorem 3] Whenever M is a McDuff II_1 factor, M is isomorphic to $M \otimes R$.*

Group and equivalence relation von Neumann algebra

The group von Neumann algebra associated to a countable group is defined as the von Neumann algebra generated by the left regular representation. The left regular representation $\lambda : G \to \mathrm{B}(\ell^2(G))$ is defined by $\lambda_g(\delta_h) = \delta_{gh}$. In this, δ_h denotes the canonical orthonormal basis of $\ell^2(G)$. The group von Neumann algebra of G is then defined as

$$\mathcal{L}(G) := \{\lambda_g \mid g \in G\}''.$$

This von Neumann algebra has a normal faithful trace, given by $\tau(x) = \langle \delta_e, x\delta_e \rangle$. $\mathcal{L}(G)$ is a factor if and only if the conjugacy classes of all non-trivial elements $g \in G$ are infinite, i.e. if the group is ICC. Examples of such groups include $\mathbb{F}_n$, S_∞ and many others. For a countable abelian group G one can show that $\mathcal{L}(G) = \mathrm{L}^\infty(\widehat{G}, \mu)$ where $\widehat{G}$ denotes the group of characters of G and μ denotes the Haar measure. This can be done using Fourier transforms. Thus, we get that $\mathcal{L}(\mathbb{Z}) \cong \mathrm{L}^\infty(\mathbb{T})$. The hyperfinite II_1 factor mentioned above can be realized as a group von Neumann algebra. Indeed, one can show that $\mathcal{L}(S_\infty) = R$. These group von Neumann algebras have proven to be very interesting. Many problems about them remain open: distinguishing between group von Neumann algebras is very hard. It is, for example, unknown whether $\mathcal{L}(\mathbb{F}_n) = \mathcal{L}(\mathbb{F}_m)$ for $n \neq m$. Obtaining isomorphism of groups from isomorphism of group von Neumann algebras can only be hoped for in the case of non-amenable groups. Indeed, it can be proven that G is amenable if and only if $\mathcal{L}(G)$ is injective, so by the result of Connes mentioned before, they are all isomorphic.

In a very similar way one can associate a von Neumann algebra to a countable equivalence relation. Let $\mathcal{R}$ be a countable equivalence relation on (X, μ) preserving μ and let $\mu^{(1)}$ be the invariant measure on $\mathcal{R}$. Denote by $[[\mathcal{R}]]$ the set of all partial automorphisms $\varphi : A \to B$ where $A, B \subset X$ are non-negligible subset of X and $x \sim_{\mathcal{R}} \varphi(x)$ for almost all $x \in A$. Consider the Hilbert space $\mathrm{L}^2(\mathcal{R}, \mu^{(1)})$ and define the left regular representation L_φ on $\mathrm{L}^2(\mathcal{R}, \mu^{(1)})$ for all $\varphi \in [[\mathcal{R}]]$ by $(L_\varphi \cdot \eta)(x, y) = \eta(\varphi^{-1}(x), y)\chi_{\mathrm{rng}\varphi}(x)$. We then have $L_\varphi L_{\varphi'} = L_{\varphi\varphi'}$ and $L_\varphi^* = L_{\varphi^{-1}}$.

We denote $\mathcal{L}(\mathcal{R}) := \{L_\varphi \mid \varphi \in [[\mathcal{R}]]\}''$. Note that $\mathrm{L}^\infty(X) \subset \mathcal{L}(\mathcal{R})$. One can show that this is in fact a Cartan subalgebra of $\mathcal{L}(\mathcal{R})$.

The standard construction $\mathcal{L}(\mathcal{R})$ can be twisted by a 2-cocycle Ω with values in $\mathbb{T}$. Let us first say what we mean by a 2-cocycle on an equivalence relation.

Definition 1.36. A 2-cocycle Ω with values in $\mathbb{T}$ is a map $\Omega : \mathcal{R}^{(2)} \to \mathbb{T}$ such that $\Omega(y, z, t)\Omega(x, z, t)^{-1}\Omega(x, y, t)\Omega(x, y, z)^{-1} = 1$ for $\mu^{(3)}$-almost-all $(x, y, z, t) \in \mathcal{R}^{(3)}$.

Using these cocycles, one defines the Ω-regular representation of an equivalence relation $\mathcal{R}$, and its twisted equivalence relation von Neumann algebra.

Definition 1.37. Let $\mathcal{R}$ be a countable equivalence relation on (X,μ) and let Ω be a 2-cocycle for $\mathcal{R}$. We define the left Ω-regular representation of $\mathcal{R}$ on $\mathrm{L}^2(\mathcal{R},\mu^{(1)})$ for all $\varphi \in [[\mathcal{R}]]$ by $(L_\varphi^\Omega \eta)(x,y) = \Omega(x,\varphi^{-1}(x),y)\eta(\varphi^{-1}(x),y)\chi_{|\mathrm{rng}(\varphi)}(x)$. We define $\mathcal{L}_\Omega(\mathcal{R}) := \{L_\varphi^\Omega \mid \varphi \in [[\mathcal{R}]]\}''$.

This construction may seem a little artificial, but the following theorem by Feldman and Moore shows its value.

Theorem 1.38. *[FM75]*

- *If $\mathcal{R} \subset X \times X$ is a type II_1 equivalence relation on the standard probability space (X,μ) then $\mathcal{L}(\mathcal{R})$ is a II_1 factor and $\mathrm{L}^\infty(X,\mu) \subset \mathcal{L}(\mathcal{R})$ is a Cartan subalgebra.*
- *Conversely, given a Cartan subalgebra A of a II_1 factor M with separable predual, then there exists a type II_1 equivalence relation $\mathcal{R}$ on a standard probability space (X,μ) and a 2-cocycle $\Omega : \mathcal{R}^{(2)} \to S^1$ such that $A \cong \mathrm{L}^\infty(X,\mu)$ and $M \cong \mathcal{L}_\Omega(\mathcal{R})$.*

Crossed product construction

Another way of building a von Neumann algebra, is from the action $G \curvearrowright (X,\mu)$ of a countable group on a standard measure space. As we said at the end of subsection 1.1.1, this translates in a natural way to an action on $\mathrm{L}^\infty(X)$. Hence this gives us an action of a countable group on a von Neumann algebra. We will define crossed product von Neumann algebras in this more general setting, where we have an action of a countable group G on a von Neumann algebra $M \subset \mathrm{B}(\mathcal{H})$.

Definition 1.39. Let $G \overset{\alpha}{\curvearrowright} M$ be an action of a countable group G on a von Neumann algebra $M \subset \mathrm{B}(\mathcal{H})$. We define the crossed product von Neumann algebra $M \rtimes G \subset \mathrm{B}(\mathcal{H} \otimes \ell^2(G))$ to be

$$M \rtimes G := \{au_g \mid a \in M, g \in G\}'',$$

where for all $\xi \otimes \delta_g \in \mathcal{H} \otimes \ell^2(G)$ we have

- $a(\xi \otimes \delta_g) = \alpha_g(a)\xi \otimes \delta_g$ for all $a \in M$,
- $u_h(\xi \otimes \delta_g) = \xi \otimes \delta_{gh^{-1}}$ for all $g \in G$.

One can show that this implies that $u_g a u_g^* = \alpha_g(a)$ for all $a \in M, g \in G$. We need a little more terminology concerning actions of groups on von Neumann algebras, to talk about the cases that are of interest to us.

Definition 1.40. Let $G \overset{\alpha}{\curvearrowright} M$ be an action of a countable group on a von Neumann algebra.

- The action is called properly outer if $ax = \alpha_g(x)a$ for all $x \in M$ forces a to be zero for all $a \in M$ and $g \in G$.
- The action is called ergodic if the only elements that are invariant under the action are scalar multiples of the identity, i.e.
$$\{x \in M \mid \alpha_g(x) = x \text{ for all } g \in G\} = \mathbb{C}1.$$

One can show that $G \curvearrowright \mathrm{L}^\infty(X,\mu)$ is properly outer if and only if $G \curvearrowright (X,\mu)$ is essentially free, and $G \curvearrowright \mathrm{L}^\infty(X,\mu)$ is ergodic if and only if $G \curvearrowright (X,\mu)$ is. If $G \curvearrowright M$ is properly outer and ergodic, $M \rtimes G$ is a factor. One can show that if M is a tracial von Neumann algebra, every element $x \in M \rtimes G$ can be written as an L^2-convergent sum $x = \sum_{g\in G} x_g u_g$ where $x_g \in M$.

Now let $G \curvearrowright (X,\mu)$ be a free, ergodic action of a countable group on a standard measure space, then $M := \mathrm{L}^\infty(X) \rtimes G$ can be divided into types as follows (see [Ta03, Theorem 1.7]): if (X,μ) is atomic then M is of type I. If there is a G-invariant probability measure on X that is equivalent to μ, M is of type II_1. If there is a G-invariant, infinite, σ-finite measure on X that is equivalent to μ, M is of type II_∞. If there is no such G-invariant σ-finite measure on X, M is of type III. In particular, whenever we have a free, ergodic, probability measure preserving action of a countable group on a standard probability space, this gives us a II_1 factor. The natural trace on $\mathrm{L}^\infty(X) \rtimes G$ is given by $\tau(fu_g) = \int_X f\delta_{e,g}d\mu$ where $\delta_{e,g}$ denotes the Kronecker delta. One can show that this is indeed a normal faithful trace. Furthermore all of these II_1 factors have a Cartan subalgebra. Indeed, one can show that $\mathrm{L}^\infty(X,\mu)$ is a maximal abelian von Neumann subalgebra that is normalized by itself and all u_g, for $g \in G$.

Finally, one can show that whenever $G \curvearrowright (X,\mu)$ is a free, ergodic, probability measure preserving action of a countable group on a standard probability space, one has $\mathrm{L}^\infty(X) \rtimes G \cong \mathcal{L}(\mathcal{R}(G \curvearrowright X))$, i.e. both constructions coincide in a logical way.

1.3.2 Rigidity results for von Neumann algebras

We can now extend Definition 1.25 with one more notion of equivalence.

Definition 1.41. Let $\Gamma \overset{\alpha}{\curvearrowright} (X,\mu)$ and $\Lambda \overset{\beta}{\curvearrowright} (Y,\nu)$ be free, ergodic, probability measure preserving actions of countable groups on standard probability spaces.

- Let $\delta : \Gamma \to \Lambda$ be a group isomorphism. A δ-conjugacy of the actions α and β is a non-singular isomorphism $\Delta : X \to Y$ such that $\Delta(g\cdot x) = \delta(g)\cdot\Delta(x)$ for all $g \in \Gamma$ and a.e. $x \in X$. If such δ and Δ exist, we say that α and β are conjugate.
- A stable orbit equivalence between the actions α and β is a non-singular isomorphism $\Delta : X_0 \to Y_0$ between non-negligible subsets $X_0 \subset X, Y_0 \subset Y$, such that Δ is an isomorphism between the restricted orbit equivalence relations $\mathcal{R}(\Gamma \curvearrowright X)_{|X_0}$ and $\mathcal{R}(\Lambda \curvearrowright Y)_{|Y_0}$. We say that α and β are stably orbit equivalent. The compression constant of Δ is defined as $c(\Delta) := \frac{\nu(Y_0)}{\mu(X_0)}$. If $\mu(X_0^c) = \nu(Y_0^c) = 0$ we say that α and β are orbit equivalent.
- A stable von Neumann equivalence, or stable W^*-equivalence between the actions α and β is a non-singular von Neumann algebra isomorphism $\Phi : p(\mathrm{L}^\infty(X) \rtimes \Gamma)p \to q(\mathrm{L}^\infty(Y) \rtimes \Lambda)q$ where p is a non-zero projection in $\mathrm{L}^\infty(X) \rtimes \Gamma$ and q is a non-zero projection in $\mathrm{L}^\infty(Y) \rtimes \Lambda$. If $p = 1$ and $q = 1$, we say that α and β are w^*-equivalent.

We already said that conjugacy implies orbit equivalence, and one can show that orbit equivalence implies w^*-equivalence. Converse results are called rigidity results. To deduce orbit equivalence from w^*-equivalence, the isomorphism of the crossed products $\Phi : \mathrm{L}^\infty(X) \rtimes \Gamma \to \mathrm{L}^\infty(Y) \rtimes \Lambda$ needs to send $\mathrm{L}^\infty(X)$ to $\mathrm{L}^\infty(Y)$. Hence results to prove w^*-superrigidity often use a 'uniqueness of Cartan' result. We briefly mention two w^*-superrigidity results.

Theorem 1.42. *[PV09, Theorem 1.2] Let $n \geq 3$ and denote by T_n the subgroup of upper triangular matrices in $\mathrm{PSL}(n,\mathbb{Z})$. Put $\Gamma = \mathrm{PSL}(n,\mathbb{Z}) *_{T_n} \mathrm{PSL}(n,\mathbb{Z})$. Then every free probability measure preserving mixing action of Γ is w^*-superrigid.*

Theorem 1.43. *[Io10, Theorem A] Let Γ be a countable ICC group which admits an infinite normal subgroup Γ_0 such that the inclusion $(\Gamma_0 \subset \Gamma)$ has the relative property (T). Let (X_0, μ_0) be a non–trivial probability space and let $\Gamma \curvearrowright (X,\mu) = (X_0,\mu_0)^\Gamma$ be the Bernoulli action. Denote $M = \mathrm{L}^\infty(X) \rtimes \Gamma$ and let $p \in M$ be a projection. Let $\Lambda \curvearrowright (Y,\nu)$ be a free ergodic p.m.p. action of a countable group Λ. Denote $N = \mathrm{L}^\infty(Y) \rtimes \Lambda$. If $N \cong pMp$ then $p = 1$, $\Gamma \cong \Lambda$ and the action $\Gamma \curvearrowright (X,\mu)$ and $\Lambda \curvearrowright (Y,\nu)$ are conjugate.*

1.3.3 Intertwining by bimodules

Whenever $\mathcal{R}_1$ and $\mathcal{R}_2$ are countable probability measure preserving equivalence relations on (X, μ) resp. (Y, ν), we have seen that if $\alpha : \mathcal{L}(\mathcal{R}_1) \to \mathcal{L}(\mathcal{R}_2)$ is an isomorphism sending $\mathrm{L}^\infty(X)$ onto $\mathrm{L}^\infty(Y)$ gives us an isomorphism of the underlying equivalence relations. In general, if $\alpha : \mathcal{L}(\mathcal{R}_1) \to \mathcal{L}(\mathcal{R}_2)$ is an isomorphism, it suffices that there exists an automorphism β of $\mathcal{L}(\mathcal{R}_2)$ such that $\beta \circ \alpha(\mathrm{L}^\infty(X)) = \mathrm{L}^\infty(Y)$ to conclude that the equivalence relations are isomorphic. Hence it would be useful to find a technique to deduce whether or not two Cartan subalgebras are conjugate by an automorphism of the bigger von Neumann algebra. To do this we need the notion of bimodules. This theory of intertwining by bimodules was developed by Popa in [Po02], [Po03].

Definition 1.44. Let M and N be two von Neumann algebras.

- A left M-module is a Hilbert space $\mathcal{H}$ equipped with a normal unital homomorphism $\pi_l : M \to \mathrm{B}(\mathcal{H})$.
- A right N-module is a Hilbert space $\mathcal{H}$ equipped with a normal unital anti-homomorphism $\pi_r : M \to \mathrm{B}(\mathcal{H})$.
- An M-N-bimodule is a Hilbert space $\mathcal{H}$ which is a left M-module and a right N-module such that the left and right actions commute. We denote it as ${}_M\mathcal{H}_N$.

We say that two M-N-bimodules $\mathcal{H}_1$ and $\mathcal{H}_2$ are isomorphic (or equivalent) if there exists a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ which intertwines the representations. Whenever we have a tracial von Neumann algebra we can define the M-dimension of any right Hilbert module $\mathcal{H}$.

Definition 1.45. Let (M, τ) be a tracial von Neumann algebra and $\mathcal{H}$ a right M-module. The M-dimension of $\mathcal{H}$ is the number $(\mathrm{Tr} \otimes \tau)(p)$ where p is any projection in $\mathrm{B}(\ell^2(\mathbb{N}) \otimes \mathrm{L}^2(M))$ such that $\mathcal{H}$ is isomorphic to $p(\ell^2(\mathbb{N}) \otimes \mathrm{L}^2(M))$. We denote this by $\dim(\mathcal{H}_M)$.

We have the following intertwining theorem.

Theorem 1.46. *[Po03, Theorem 2.1] Let (M, τ) be a tracial von Neumann algebra, $p \in M$ a non-zero projection, and A, B two von Neumann subalgebras of pMp or M respectively. The following conditions are equivalent.*

- *There is no sequence (u_n) of unitary elements in A such that, for every $x, y \in M$ we have $\lim_n \|E_B(x^* u_n y)\| = 0$.*

- *There exists a non-zero* A*-*B*-subbimodule* $\mathcal{H}$ *of* $p\mathrm{L}^2(M)$ *such that* $\dim(\mathcal{H}_B) < +\infty$.
- *There exists an integer* $n \geq 1$*, a projection* $q \in \mathrm{M}_n(M) \otimes B$*, a partial isometry* $v \in \mathrm{M}_{1,n}(\mathbb{C}) \otimes pM$ *and a normal unital homomorphism* $\psi : A \to q(\mathrm{M}_n(\mathbb{C}) \otimes B)q$ *such that* $v^*v \leq q$ *and* $xv = v\psi(x)$ *for every* $x \in A$.

If any of these conditions is satisfied, we write $A \prec_M B$*. We say* A *embeds into* B *inside* M*.*

For Cartan subalgebras in a II_1 factor, even more can be said.

Theorem 1.47. *[Po02, Theorem A.1] Let* A *and* B *be Cartan subalgebras in a* II_1 *factor* M *such that* $A \prec_M B$*. Then there exists a unitary element* $u \in M$ *such that* $uAu^* = B$*.*

1.4 The fundamental group

In this section we discuss the fundamental group for von Neumann algebras and equivalence relations. It was introduced by Murray and von Neumann for type II_1 factors in [MvN43]. This notion is not related to the fundamental group of a topological space, but instead is a subgroup of the positive real numbers.

Definition 1.48. Let (M, τ) be a II_1 factor and $\mathcal{R}$ be a II_1 equivalence relation on the standard probability space (X, μ).

- The fundamental group $\mathcal{F}(M)$ of M is the set
$$\mathcal{F}(M) := \left\{ \frac{\tau(p)}{\tau(q)} \mid p, q \text{ projections in } M, pMp \cong qMq \right\}.$$
- The fundamental group $\mathcal{F}(\mathcal{R})$ of $\mathcal{R}$ is the set
$$\mathcal{F}(\mathcal{R}) := \left\{ \frac{\mu(U)}{\mu(V)} \mid U, V \text{ non-negligible subsets of } X, \mathcal{R}_{|U} \cong \mathcal{R}_{|V} \right\}.$$

Note that pMp is again a II_1 factor, and $\mathcal{R}_{|U}$ is a II_1 equivalence relation. One can show that this is indeed a multiplicative subgroup of $\mathbb{R}^*_+$. Furthermore one also has that $\mathcal{F}(\mathcal{R}) \subset \mathcal{F}(\mathcal{L}(\mathcal{R}))$. This inclusion can be strict, and we will in fact give examples where this is the case (see chapter 5). On the other hand if for all $p, q \in \mathrm{L}^\infty(X)$ and for every isomorphism $\varphi : p\mathcal{L}(\mathcal{R})p \to q\mathcal{L}(\mathcal{R})q$ we have that $p\mathrm{L}^\infty(X)$ is mapped to $uq\mathrm{L}^\infty(X)u^*$ for some unitary $u \in q\mathcal{L}(\mathcal{R})q$, we have $\mathcal{F}(\mathcal{R}) = \mathcal{F}(\mathcal{L}(\mathcal{R}))$.

For a II_1 equivalence relation $\mathcal{R}$ on a standard probability space (X, μ), it is not hard to see that $\mathcal{F}(\mathcal{R}) = \{c(\Delta) \mid \Delta \text{ is a sOE between } G \curvearrowright X \text{ and } G \curvearrowright X\}$.

Calculating this fundamental group is far from easy: Murray and von Neumann only were able to show that the fundamental group of the hyperfinite II_1 factor is $\mathbb{R}_+^*$. Note that this implies that the fundamental group of any McDuff II_1 factor is also $\mathbb{R}_+^*$. In some cases one can make sure certain elements are definitely contained in the fundamental group. Before we look into this, we need the notion of a commensurator.

Definition 1.49. Let Γ be a countable group.

- Consider the group $\Omega(\Gamma)$ of isomorphisms $\delta : \Gamma_1 \to \Gamma_1'$ such that both $[\Gamma : \Gamma_1]$ and $[\Gamma : \Gamma_1']$ are finite. Two such isomorphisms $\varphi_1 : \Gamma_1 \to \Gamma_1'$ and $\varphi_2 : \Gamma_2 \to \Gamma_2'$ are called equivalent (not. $\varphi_1 \sim \varphi_2$) whenever there is a finite index $\Gamma_0 < \Gamma$ on which both φ_1 and φ_2 are defined, and $(\varphi_1)_{|\Gamma_0} = (\varphi_2)_{|\Gamma_0}$. The set $\Omega(\Gamma)/\sim$ is called the abstract commensurator of Γ, denoted $\mathrm{Comm}(\Gamma)$.
- Let $\Gamma \overset{\alpha}{\curvearrowright} X$ be a free, ergodic, p.m.p. action. The abstract commensurator of $\Gamma \overset{\alpha}{\curvearrowright} X$, denoted $\mathrm{Comm}(\Gamma \overset{\alpha}{\curvearrowright} X)$, is the set of tuplets $(\Gamma_1, \Gamma_2, X_1, X_2, \Delta)$ such that
  - $[\Gamma : \Gamma_1] < \infty, [\Gamma : \Gamma_2] < \infty$,
  - $\Gamma \overset{\alpha}{\curvearrowright} X$ is induced from both $\Gamma_1 \overset{\alpha_{|\Gamma_1}}{\curvearrowright} X_1$ and $\Gamma_2 \overset{\alpha_{|\Gamma_2}}{\curvearrowright} X_2$,
  - $\alpha_{|\Gamma_1}$ and $\alpha_{|\Gamma_2}$ are conjugate through Δ,

  where two such tuplets are the same if they coincide on a finite index subgroup.

One can show that if Γ is an ICC group, the automorphisms of Γ embed injectively into $\mathrm{Comm}(\Gamma)$. The following result gives us a way to make actions that have certain elements in the fundamental group of the associated equivalence relation.

Theorem 1.50. *Let $\Gamma \curvearrowright X$ free, ergodic, p.m.p. Whenever*

$$(\Gamma_1, \Gamma_2, X_1, X_2, \Delta) \in \mathrm{Comm}(\Gamma \curvearrowright X),$$

we have that $\frac{[\Gamma:\Gamma_1]}{[\Gamma:\Gamma_2]} \in \mathcal{F}(\mathcal{R}(\Gamma \curvearrowright X))$.

Proof. As we remarked earlier,

$$\mathcal{F}(\mathcal{R}(\Gamma \curvearrowright X)) = \{c(\Delta) \mid \Delta : X \to X \text{ a self-sOE}\}.$$

Suppose $(\Gamma_1, \Gamma_2, X_1, X_2, \Delta) \in \mathrm{Comm}(\Gamma \curvearrowright X)$. Let $\Delta : X_1 \to X_2$ be the measure space isomorphism that conjugates α_1 with α_2. Let $\Delta_1 : X \to X_1$ resp. $\Delta_2 : X \to X_2$ be the natural stable orbit equivalences we get from the fact that $\Gamma \curvearrowright X$ is induced from both $\Gamma_1 \curvearrowright X_1$ and $\Gamma_2 \curvearrowright X_2$. Set $\psi : X \to X : \Delta_2^{-1} \circ \Delta_0 \circ \Delta_1$. It is immediately clear that this is a self-sOE for $\Gamma \curvearrowright X$, with compression constant $c(\psi) = \frac{c(\Delta_1)}{c(\Delta_2)} = \frac{[\Gamma:\Gamma_2]}{[\Gamma:\Gamma_1]}$. □

So if we could build an action such that its commensurator is non-trivial, this could be a way of putting elements in the fundamental group. We will come back to this in section 2.2. Another way to obtain elements in the fundamental group of an equivalence relation, is showing that the group acting on the standard measure space has finite normal subgroups.

Proposition 1.51. *Let $\Gamma \curvearrowright (X, \mu)$ be a free, ergodic, probability measure preserving action of a countable group on a standard measure space. Whenever Γ has finite normal subgroups N_1, N_2 such that $\Gamma \curvearrowright X$ is conjugate to $\Gamma/N_i \curvearrowright X/N_i$ (with normalized measure), then $\frac{|N_1|}{|N_2|} \in \mathcal{F}(\mathcal{R}(\Gamma \curvearrowright X))$.*

Proof. This follows from the fact that $\Gamma \curvearrowright X$ is stably orbit equivalent to $\Gamma/N_i \curvearrowright X/N_i$ with compression constant $|N_i|$. □

So from this one can see that there are two ways in which one can 'see' elements in the fundamental group: the action can be induced from another action, or the group acting can have finite normal subgroups. This is not necessarily the whole fundamental group.

1.5 Property (T)

Certain group properties make dealing with orbit equivalence relations, group von Neumann algebras and group measure space constructions 'easier'. One of these properties we already briefly mentioned: a group is amenable if it admits a left-invariant mean. Another property we will need in chapter 5 is the Haagerup property. We will give its definition and examples in §5.1. A third property, that is used more often in this thesis, is Kazhdan's property (T).

Definition 1.52. Let G be a locally compact second countable group.

- A (unitary) representation of G on a Hilbert space $\mathcal{H}$ is a group morphism $\pi : G \to \mathcal{U}(\mathcal{H})$ that is strongly continuous.
- A representation $(\pi, \mathcal{H})$ of G is said to have a (Q, ϵ) invariant vector $\xi \in \mathcal{H}$, where $\epsilon > 0$ and $Q \subset G$, if $\sup_{x \in Q} \|\pi(x)\xi - \xi\| < \epsilon\|\xi\|$.
- A representation $(\pi, \mathcal{H})$ of G is said to have almost invariant vectors if it has (Q, ϵ)-invariant vectors for every compact set Q of G and every $\epsilon > 0$.
- G has property (T) if every representation admitting almost invariant vectors has a non-zero invariant vector.

Equivalently, G has property (T) if there exists $\epsilon > 0$ and a compact $K \subset G$ such that every unitary representation of G that has a (K, ϵ)-invariant unit vector, has a non-zero invariant vector. Such a pair (K, ϵ) is called a Kazhdan pair for G.

One can show that property (T) and amenability are 'far from each other' in the sense that any amenable group that has property (T) has to be compact. Various equivalent notions of this definition are known, and for a detailed survey on property (T) we refer to [BHV07]. We only give some examples of groups with property (T).

Example 1.53.
- For any local field $\mathbb{K}$, the group $\mathrm{SL}_n(\mathbb{K})$ had property (T) whenever $n \geq 3$.
- The semi-direct product $\mathrm{SL}_n(\mathbb{K}) \ltimes \mathbb{K}^n$ also has property (T) for $n \geq 3$.
- The semi-direct product $\mathrm{SL}_n(\mathbb{K}) \ltimes M_{n,m}(\mathbb{K})$ has property (T) for $n \geq 3$ and any $m \in \mathbb{N}$.
- The symplectic group $\mathrm{Sp}_{2n}(\mathbb{K})$ has property (T) for $n \geq 2$.
- If Γ is a lattice in G then G has property (T) if and only if Γ has property (T).

Non-examples include $\mathbb{R}^n$, $\mathbb{Z}^n$, $\mathbb{F}_n$ and $\mathrm{SL}_2(\mathbb{Z})$. One can show that if G has property (T) and $(\pi, \mathcal{H})$ is a representation with almost invariant vectors ξ_n, then there are invariant vectors ξ 'close to' the almost invariant vectors in the following sense.

Proposition 1.54. *[BHV07, Proposition 1.1.9] Let G be a locally compact group, let (Q, ϵ) be a Kazhdan pair for G, and let $\delta > 0$. Then, for every unitary representation $(\pi, \mathcal{H})$ of G and every $(Q, \delta\epsilon)$-invariant vector ξ, we have*

$$\|\xi - P\xi\| \leq \delta\|\xi\|,$$

where $P : \mathcal{H} \to \mathcal{H}^G$ is the orthogonal projection on the subspace $\mathcal{H}^G$ of all G-invariant vectors in $\mathcal{H}$.

One can also define property (T) for pairs of groups.

Definition 1.55. Let G be a topological group and H be a closed subgroup of G. The pair (G, H) is said to have relative property (T) (or the inclusion $H \subset G$ is called rigid) if, whenever a unitary representation $(\pi, \mathcal{H})$ of G has almost invariant vectors, it actually has a non-zero H-invariant vector.

It is immediately clear that whenever H has property (T) and $H < G$ closed, then $H \subset G$ is rigid. Another well known example is the fact that $(\mathrm{SL}_2(\mathbb{K}) \ltimes \mathbb{K}^2, \mathbb{K}^2)$ has relative property (T) for every local field $\mathbb{K}$.

The notion of property (T) can be defined for measured groupoids in general, see [A-D03]. We only need the concept of property (T) for non-singular actions of locally compact second countable groups on standard measure spaces. We give this definition in an operator algebra framework, and recall the notion of property (T) for actions of locally compact second countable groups on von Neumann algebras.

Given a von Neumann algebra $M \subset \mathrm{B}(\mathcal{H})$ and a Hilbert space $\mathcal{K}$, define the W^*-module $M \overline{\otimes} \mathcal{K}$ as $\{T \in \mathrm{B}(\mathcal{H}, \mathcal{H} \otimes \mathcal{K}) \mid Tx = (x \otimes 1)T, \forall x \in M'\}$. Note that this is an $(M \overline{\otimes} \mathrm{B}(\mathcal{K}))$-$M$-bimodule, and that it inherits the strong*-topology from $\mathrm{B}(\mathcal{H}, \mathcal{H} \otimes \mathcal{K})$. Equip $\mathrm{Aut}(M)$ with the Polish topology that makes $\mathrm{Aut}(M) \to M_* : \alpha \mapsto \omega \circ \alpha$ continuous for all $\omega \in M_*$, then an action of a locally compact second countable group G on M is a continuous group morphism $\alpha : G \to \mathrm{Aut}(M)$. To define property (T) in this setting, we need the notion of a 1-cocycle for an action of a locally compact second countable group on a von Neumann algebra (in the next section, we will also define 1-cocycles for equivalence relations and actions on standard measure spaces).

Definition 1.56. Let $G \overset{\alpha}{\curvearrowright} M$ be an action of a locally compact second countable group G on a von Neumann algebra M.

- A 1-cocycle for α with values in the unitary group $\mathcal{U}(\mathcal{K})$ of a Hilbert space $\mathcal{K}$ is a strongly continuous map $c : G \to \mathcal{U}(M \overline{\otimes} \mathrm{B}(\mathcal{K}))$ satisfying $c(gh) = c(g)(\alpha_g \otimes \mathrm{id})(c(h))$ for all $g, h \in G$.
- A unit invariant vector of a 1-cocycle c of α is an element $\xi \in M \overline{\otimes} \mathcal{K}$ such that $\xi^*\xi = 1$ and $c(g)(\alpha_g \otimes \mathrm{id})(\xi) = \xi$ for all $g \in G$.
- A sequence of almost invariant unit vectors of a 1-cocycle c of α is a sequence $\xi_n \in M \overline{\otimes} \mathcal{K}$ such that $\xi_n^*\xi_n = 1$ for all n and $c(g)(\alpha_g \otimes \mathrm{id})(\xi_n) - \xi_n \to 0$ $*$-strongly and uniformly on compact subsets of G.

- α is said to have property (T) if every 1-cocycle as above and admitting a sequence of almost invariant unit vectors, admits a unit invariant vector.

[PV08b, Proposition 3.2] states that for probability measure preserving actions $G \curvearrowright (X, \mu)$ of locally compact second countable groups of standard probability spaces, $G \curvearrowright (X, \mu)$ has property (T) if and only if G has property (T) (in fact there it is shown in a more general von Neumann algebra setup).

1.6 Cocycle superrigidity

We start by defining the notion of a 1-cocycle of an equivalence relation and of an action of a locally compact second countable group on a standard measure space. To do this, we need the notion of a Polish group of finite type.

Definition 1.57. [Po05, Definition 2.5] A Polish group is of finite type if it can be realized as a closed subgroup of the unitary group of some II_1 factor with separable predual.

All countable and all second countable compact groups are Polish groups of finite type.

Definition 1.58. Let $\mathcal{R}$ be a measure preserving countable equivalence relation on a standard measure space, and $G \curvearrowright (X, \mu)$ an action of a locally compact second countable group on a standard measure space.

- A 1-cocycle for $G \curvearrowright (X, \mu)$ with values in a Polish group of finite type $\mathcal{U}$ is a continuous map $\omega : G \times X \to \mathcal{U}$ such that $\omega(gh, x) = \omega(g, x)\omega(h, g \cdot x)$ for all $g, h \in G$ and almost all $x \in X$.
- A 1-cocycle for $\mathcal{R}$ with values in a Polish group of finite type $\mathcal{U}$ is a continuous map $\omega : \mathcal{R} \to \mathcal{U}$ such that $\omega(x, z) = \omega(x, y)\omega(y, z)$ for almost all $(x, y, z) \in \mathcal{R}^{(2)}$.

Whenever we have a stable orbit equivalence $\Delta : U \to V$ between free actions $G \curvearrowright X$ and $H \curvearrowright Y$, one can associate to this the 'Zimmer 1-cocycle'. Indeed, let $\omega : X \times G \to H$ be the measurable map such that $\Delta(g \cdot x) = \omega(g, x)\Delta(x)$ for almost all $x \in X$ and all $g \in G$. One checks that this is indeed a 1-cocycle.

Many results in the study of countable equivalence relations and von Neumann algebras rely on some form of 'cocycle superrigidity'.

Definition 1.59. [Po05, Definition 2.5] A non-singular action $G \curvearrowright (X,\mu)$ of a locally compact second countable group G on a standard measure space (X,μ) is called $\mathcal{U}_{fin}$-cocycle superrigid if every 1-cocycle for the action $G \curvearrowright (X,\mu)$ with values in a Polish group of finite type $\mathcal{G}$ is cohomologous to a continuous group morphism $G \to \mathcal{G}$.

Note that continuous group morphisms $G \to \mathcal{G}$ can be seen as 1-cocycles that are independent of X. If one has such a cocycle superrigidity result for a class of groups or actions, this makes studying the self-stable orbit equivalences easier and gives us a way to obtain OE superrigidity results. In section 3.2 we will say a little more on such cocycle superrigidity results, but for now we limit ourselves to proving the following slightly different version of [Po05, Proposition 3.6 (2)] (see also [Fu06, Lemma 3.5]).

Lemma 1.60. *Let $G \curvearrowright (X,\mu)$ be a non-singular action of a countable group G on a standard measure space (X,μ). Let $\omega : G \times X \to \mathcal{G}$ be a 1-cocycle with values in the Polish group $\mathcal{G}$ with a bi-invariant metric. Let $H < G$ be a subgroup and assume that $\omega(h,x) = \delta(h)$ for all $h \in H$ and a.e. $x \in X$, where $\delta : H \to \mathcal{G}$ is a group morphism. For any $g_0 \in G$ such that the diagonal action $H_0 = H \cap g_0^{-1} H g_0 \curvearrowright X \times X$ is ergodic, $x \mapsto \omega(g_0,x)$ is essentially constant.*

Proof. For any $h \in H_0$ denote $\alpha(h) = g_0 h g_0^{-1}$ and remark that $h, \alpha(h) \in H$. For all $h \in H_0$ and a.e. $x \in X$ we have

$$\begin{aligned}
\delta(\alpha(h))\omega(g_0,x) &= \omega(\alpha(h), g_0 \cdot x)\omega(g_0,x) \\
&= \omega(\alpha(h)g_0, x) \\
&= \omega(g_0 h, x) \\
&= \omega(g_0, h \cdot x)\omega(h,x) \\
&= \omega(g_0, h \cdot x)\delta(h)\ ,
\end{aligned}$$

so $\omega(g_0, h\cdot x) = \delta(\alpha(h))\omega(g_0,x)\delta(h^{-1})$. Now consider the map $\varphi : X \times X \to \mathbb{R} : (x,y) \mapsto d(\omega(g_0,x), \omega(g_0,y))$. Then φ is essentially invariant under the diagonal action $H_0 \curvearrowright X \times X$, as the metric is bi-invariant. But as $H_0 \curvearrowright X \times X$ is ergodic, φ has to be essentially constant, hence 0. This proves that $x \mapsto \omega(g_0,x)$ is essentially constant. □

Chapter 2

II_1 factors with two non-conjugate Cartan subalgebras

As was said before, whenever a II_1 factor contains a Cartan subalgebra, there exists an equivalence relation and a 2-cocycle such that the twisted equivalence relation von Neumann algebra is the original II_1 factor. In particular this means that if Cartan subalgebras are unique up to unitary conjugacy, then any automorphism of the II_1 factor has to map the Cartan subalgebra to itself if the 2-cocycle is trivial. In this case, the fundamental group of the II_1 factor is exactly the same as that of the associated equivalence relation. However, as we are interested in situations where the fundamental groups are different, it makes sense to look for situations where there are more than one Cartan subalgebras.

Initially, the intention was to look for examples that were sort of opposite to the examples given in chapter 5. There we give examples of group measure space constructions such that the II_1 factor has $\mathbb{R}_0^+$ as fundamental group, while the equivalence relation can have any group in a 'large' class of groups as fundamental group. However, giving examples of this opposite scenario, where the fundamental group of the equivalence relation would be trivial but the II_1 factor has non-trivial fundamental group, proved to be too ambitious. A natural start for this would be to look for examples of II_1 factors with multiple Cartan subalgebras, such that the associated II_1 equivalence relations could be easily described. Then we might obtain situations where this second equivalence relation has strictly smaller fundamental group than the original one. However, in studying these equivalence relations associated to Cartan subalgebras, the interesting cases were the ones where the second Cartan subalgebra was twisted by a non-trivial cocycle. In this case, the fundamental group of the equivalence relation associated to the twisted Cartan subalgebra could be bigger than the fundamental group of the original II_1 factor. In chapters 3 and 4 we study such an example.

In section 1, we give a lemma that gives us a technique to find a second Cartan subalgebra in a special setting. In section 2 we will give two examples that satisfy the conditions of this lemma. However neither of these examples will prove interesting, in that in both cases there will be no difference in fundamental group between the two equivalence relations. In section 3 we give another example which will satisfy the conditions of the lemma. This final example will be more interesting, and we study it in chapters 4 and 5. For a more in depth study of situations with many non-conjugate Cartan subalgebras, see [SV11] or An Speelman's PhD Thesis [Sp13].

2.1 Finding a second Cartan subalgebra

The first example of a II_1 factor with two Cartan subalgebras that are not conjugate by an automorphism was given by Connes and Jones in [CJ82]. In [OP08, §7], Ozawa and Popa gave more explicit examples of this phenomenon. In fact, a II_1 factor can have uncountably many non-conjugate Cartan subalgebras (see [Po86], [Po90], [Po06] and [SV11]). The construction in the lemma originates from [PV09, Example 5.8] and is closely related to [SV11, Lemma 6].

Lemma 2.1. *Let $\mathcal{Z}$ be a compact abelian group and $Z < \mathcal{Z}$ a countable subgroup. Let $Z_0 < Z$ be an infinite subgroup. Assume that Z acts on $\mathcal{Z}$ by translation. Let Γ be a countable group that acts on $\mathcal{Z}$ by continuous group automorphisms $(\alpha_g)_{g\in\Gamma}$ preserving Z and Z_0. Define*

$$M := \mathrm{L}^\infty(\mathcal{Z}) \rtimes (Z \rtimes \Gamma)\ .$$

Denote $\mathcal{Z}_0 := \overline{Z_0}$. Assume that for all $g \in \Gamma : \{z - \alpha_g(z) \mid z \in Z_0\}$ is either infinite or trivial. Denote $\Gamma_0 := \{g \in \Gamma \mid \alpha_g(z) = z \text{ for all } z \in Z_0\}$. Assume that $\{x \in \frac{\mathcal{Z}}{\mathcal{Z}_0} \mid \alpha_g(x) = x\}$ has infinite index in $\frac{\mathcal{Z}}{\mathcal{Z}_0}$ for all $g \in \Gamma_0\backslash\{e\}$.

Assume finally that $Z \cap \mathcal{Z}_0 = Z_0$. Then $A := \mathcal{L}(Z_0) \overline{\otimes} \mathrm{L}^\infty(\frac{\mathcal{Z}}{\mathcal{Z}_0}) = \mathrm{L}^\infty(\widehat{Z_0} \times \frac{\mathcal{Z}}{\mathcal{Z}_0})$ is a Cartan subalgebra of M and the induced equivalence relation on $\widehat{Z_0} \times \frac{\mathcal{Z}}{\mathcal{Z}_0}$ is given by the action $(\widehat{\mathcal{Z}_0} \times \frac{Z}{Z_0}) \rtimes \Gamma \curvearrowright \widehat{Z_0} \times \frac{\mathcal{Z}}{\mathcal{Z}_0}$, where $\widehat{\mathcal{Z}_0} \times \frac{Z}{Z_0}$ acts on $\widehat{Z_0} \times \frac{\mathcal{Z}}{\mathcal{Z}_0}$ by translation and Γ acts on both components in the natural way.

Proof. First look at $\mathcal{L}(Z_0)' \cap M$. Take $(t,e) \in Z_0 \times \{e\}$ and $x \in \mathcal{L}(Z_0)' \cap M$. Write $x = \sum_{(s,g)\in Z\rtimes\Gamma} a_{(s,g)}u_{(s,g)}$. We have $u_{(t,e)}x = xu_{(t,e)}$ so

$$\sum_{(s,g)\in Z\rtimes\Gamma} \alpha_{(t,e)}(a_{(s,g)})u_{(s+t,g)} = \sum_{(s,g)\in Z\rtimes\Gamma} a_{(s,g)}u_{(s+g\cdot t,g)}.$$

But then

$$\sum_{(s,g)\in Z\rtimes\Gamma} \alpha_{(t,e)}(a_{(s-t,g)})u_{(s,g)} = \sum_{(s,g)\in Z\rtimes\Gamma} a_{(s-g\cdot t,g)}u_{(s,g)}.$$

So we see that for all $t \in Z_0$ and for all $(s,g) \in Z \rtimes \Gamma$:

$$\alpha_{(t,e)}(a_{(s-t,g)}) = a_{(s-g\cdot t,g)}, \text{ and hence } \alpha_{(t,e)}(a_{(s,g)}) = a_{(s+t-g\cdot t,g)}.$$

Now fix $g \in \Gamma$ and assume $\{t - g\cdot t \mid t \in Z_0\}$ is infinite. One can see that in that case $a_{(s,g)} = 0$. Indeed, otherwise $\|x\|_2^2 = \sum_{(s,g)\in Z\rtimes\Gamma} \|a_{(s,g)}\|^2 = \infty$. Now suppose $g \in \Gamma_0$. In that case we get that $a_{(s,g)} \in \mathrm{L}^\infty(\frac{\mathcal{Z}}{\mathcal{Z}_0})$. So we find

$$\mathcal{L}(Z_0)' \cap M = \mathrm{L}^\infty(\frac{\mathcal{Z}}{\mathcal{Z}_0}) \rtimes (Z \rtimes \Gamma_0). \tag{2.1}$$

Let $x \in \mathrm{L}^\infty(\frac{\mathcal{Z}}{\mathcal{Z}_0})' \cap \Big(\mathrm{L}^\infty(\frac{\mathcal{Z}}{\mathcal{Z}_0}) \rtimes (Z \rtimes \Gamma_0)\Big)$. Write $x = \sum_{(s,g)\in Z\rtimes\Gamma_0} a_{(s,g)}u_{(s,g)}$ with $a_{s,g} \in \mathrm{L}^\infty\left(\frac{\mathcal{Z}}{\mathcal{Z}_0}\right)$. Then for all $f \in \mathrm{L}^\infty\left(\frac{\mathcal{Z}}{\mathcal{Z}_0}\right)$ we have

$$\sum_{(s,g)\in Z\rtimes\Gamma_0} fa_{(s,g)}u_{(s,g)} = \sum_{(s,g)\in Z\rtimes\Gamma_0} f((sZ_0,g)^{-1}\cdot)a_{(s,g)}u_{(s,g)}.$$

So for all $(s,g) \in Z \rtimes \Gamma_0$ we get that $fa_{(s,g)} = f((sZ_0,g)^{-1}\cdot)a_{(s,g)}$ for all f. By assumption $\{x \in \frac{\mathcal{Z}}{\mathcal{Z}_0} \mid g\cdot x = x\}$ has infinite index in $\frac{\mathcal{Z}}{\mathcal{Z}_0}$ for all $g \in \Gamma_0\backslash\{e\}$. By [SV11, Lemma 5] we find that $\frac{Z}{Z_0} \rtimes \Gamma_0 \curvearrowright \frac{\mathcal{Z}}{\mathcal{Z}_0}$ is essentially free. So if $(s,g) \notin Z_0$ then $a_{(s,g)} = 0$. This implies that

$$\left(\mathrm{L}^\infty\left(\frac{\mathcal{Z}}{\mathcal{Z}_0}\right)\right)' \cap \left(\mathrm{L}^\infty\left(\frac{\mathcal{Z}}{\mathcal{Z}_0}\right) \rtimes (Z \rtimes \Gamma_0)\right) = \mathrm{L}^\infty\left(\frac{\mathcal{Z}}{\mathcal{Z}_0}\right) \rtimes Z_0.$$

In combination with 2.1, we get that A is maximal abelian.

For any $\omega \in \hat{\mathcal{Z}}$ we define the unitary $U_\omega \in \mathrm{L}^\infty(\mathcal{Z})$ by $U_\omega(x) = \omega(x)$. We check that A is normalized by $\{u_s \mid s \in Z\}$, $\{u_g \mid g \in \Gamma\}$ and $\{U_\omega \mid \omega \in \widehat{\mathcal{Z}}\}$. Once we have this, we know A is regular in M and hence A is a Cartan subalgebra of M.

First consider $\{u_s \mid s \in Z\}$. It is clear that these commute with $\mathcal{L}(Z_0)$, so we only need to look at $\mathrm{L}^\infty(\frac{\mathcal{Z}}{\mathcal{Z}_0})$. Take a a Z_0 invariant function and $s \in Z$. Then for any $t \in Z_0$ we have $t \cdot u_s a u_s^* = t \cdot a(\cdot - s) = a(\cdot - s) = u_s a u_s^*$. This implies that A is normalized by $\{u_s \mid s \in Z\}$.

Next let $g \in \Gamma$. Take $s \in Z_0$, then one easily sees $u_g u_s u_g^* = u_{g\cdot s}$ and as Z_0 is globally Γ invariant, this is again an element in $\mathcal{L}(Z_0)$. For an element in $\mathrm{L}^\infty(\frac{\mathcal{Z}}{\mathcal{Z}_0})$, the same reasoning as above applies.

Finally take $\omega \in \widehat{\mathcal{Z}}$ then $U_\omega \in \mathrm{L}^\infty(\mathcal{Z})$. This commutes with $\mathrm{L}^\infty(\frac{\mathcal{Z}}{\mathcal{Z}_0})$. Let $s \in Z_0$ then $U_\omega u_s U_\omega^* = U_\omega U_\omega^*(\cdot - s) u_s = \omega(s) u_s \in \mathcal{L}(Z_0)$.

Now that we know A is a Cartan subalgebra of M, we can take a look at the induced equivalence relation. We check how $\mathrm{Ad}\, u_s$, $\mathrm{Ad}\, u_g$ and $\mathrm{Ad}\, U_\omega$ for $s \in Z, g \in \Gamma, \omega \in \widehat{\mathcal{Z}}$ act on A and how they interact.

First take $\mathrm{Ad}\, u_s$. Clearly $\mathrm{Ad}\, u_s$ does not act on $\mathcal{L}(Z_0)$, but only on $\mathrm{L}^\infty(\frac{\mathcal{Z}}{\mathcal{Z}_0})$. On the latter it acts by translating by the class of s in $\frac{Z}{Z_0}$. Next let $\mathrm{Ad}\, u_g$. This acts both on $\mathcal{L}(Z_0)$ and $\mathrm{L}^\infty(\frac{\mathcal{Z}}{\mathcal{Z}_0})$ in the usual way. Finally consider $\mathrm{Ad}\, U_\omega$. This only acts on $\mathrm{L}^\infty(\widehat{Z_0})$ by restricting to a character on Z_0 and then translating by this character.

We found actions of $\frac{Z}{Z_0}, \Gamma$ and $\widehat{\mathcal{Z}_0}$ on $\mathrm{L}^\infty(\widehat{Z_0} \times \frac{\mathcal{Z}}{\mathcal{Z}_0})$. Remark that for all $g \in \Gamma, s \in \frac{Z}{Z_0}, \omega \in \widehat{\mathcal{Z}_0}, a \in \mathrm{L}^\infty(\frac{\mathcal{Z}}{\mathcal{Z}_0})$ and $a_s \in \mathrm{L}^\infty(\widehat{Z_0})$ we have

$$\begin{aligned}
\mathrm{Ad}\, u_g(\mathrm{Ad}\, u_s(a)) &= u_{(e,g)} u_{(s,e)} a u_{(s,e)}^* u_{(e,g)}^* \\
&= u_{(g\cdot s, g)} a u_{(s^{-1}, g^{-1})} \\
&= u_{(g\cdot s, e)} u_{(e,g)} a u_{(e,g)}^* u_{(g\cdot s, e)}^* \\
&= \mathrm{Ad}\, u_{g\cdot s}(\mathrm{Ad}\, u_g(a)),
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Ad}\, U_\omega(\mathrm{Ad}\, u_g(a_s)) &= U_\omega u_g a_s u_g^* U_\omega^* \\
&= U_\omega a_{g\cdot s} U_\omega^* \\
&= \omega(g\cdot s) a_{g\cdot s} \\
&= u_g \omega(g \cdot s) a_s u_g^* \\
&= u_g U_\omega(g^{-1} \cdot) a_s U_\omega^*(g^{-1} \cdot) u_g^* \\
&= \mathrm{Ad}\, u_g(\mathrm{Ad}\, U_\omega(g^{-1} \cdot)(a_s)).
\end{aligned}$$

It follows that the induced equivalence relation is given by the orbits of the action

$$\left(\widehat{\mathcal{Z}_0} \times \frac{Z}{Z_0}\right) \rtimes \Gamma \curvearrowright \widehat{Z_0} \times \frac{\mathcal{Z}}{\mathcal{Z}_0},$$

where $\widehat{\mathcal{Z}_0} \times \frac{Z}{Z_0}$ acts on $\widehat{Z_0} \times \frac{\mathcal{Z}}{\mathcal{Z}_0}$ by translation and Γ acts in the natural way. $\square$

2.2 Applications of the lemma

As was said in the introduction to this chapter, the intended application of the lemma was the construction of II_1 factors such that the fundamental group of the equivalence relation associated to the second Cartan subalgebra would be different from the fundamental group of the II_1 factor. As was remarked in section 1.4, there are some ways to assure some elements will be contained in the fundamental group, by choosing specific kinds of groups or actions. Before we give applications of the above lemma, we recall what is meant by a profinite action.

Profinite actions

Before we can say what we mean by a profinite action, we need to explain what we mean by an inverse limit of actions. So let $G \overset{\alpha_n}{\curvearrowright} (X_n, \mu_n)$ be a sequence of measure preserving actions of a countable group G on standard probability spaces (X_n, μ_n). Assume α_n is a quotient of α_{n+1} for all n. Denote by $p_n : (X_{n+1}, \mu_{n+1}) \to (X_n, \mu_n)$ a measurable, measure preserving, onto map such that $p_n(g \cdot x) = g \cdot p_n(x)$ for almost all $x \in X_{n+1}$ and all $g \in G$. Set

$$X := \{(x_n)_n \mid x_n \in X_n, p_n(x_{n+1}) = x_n \text{ for all } n\}.$$

Let $q_n : X \to X_n$ be the projection on the n^{th} component, i.e. $q_n((x_m)_m) = x_n$. In this setting there is a unique probability measure μ on X such that q_n is measurable and measure preserving for all n. We let G act on X by setting $\alpha_g(x_n)_n = (g \cdot x_n)_n$. That way α is μ-preserving and we can view α_n as a quotient of α through q_n. Then α is called the inverse limit of α_n and we denote

$$(X, \mu) := \varprojlim (X_n, \mu_n) \text{ and } \alpha := \varprojlim \alpha_n.$$

Definition 2.2. A measure preserving action $G \overset{\alpha}{\curvearrowright} (X, \mu)$ is a profinite action if $\alpha = \varprojlim \alpha_n$ for a sequence of measure preserving actions $G \overset{\alpha_n}{\curvearrowright} (X_n, \mu_n)$ where X_n is a finite set.

For more details, examples and properties of profinite actions, see [Io08, §1]. We limit ourselves to remarking that any ergodic profinite action $G \curvearrowright (X, \mu) = \varprojlim (X_n, \mu_n)$ arises as $G \curvearrowright \varprojlim G/G_n$ where $\{G_n\}$ is a descending chain of finite index subgroups of G and G acts on the right cosets G/G_n by left translation. Indeed, as α is ergodic, G acts transitively on X_n, so $X_n = Ga_n$ for some $a_n \in X_n$ and this for all n. Set $G_n := \{g \in G \mid ga_n = a_n\}$. Then $G_{n+1} \subset G_n$ and the map $\psi : X \to \varprojlim G/G_n$ given by $\psi((x_n)_n) = (g_n G_n)_n$, where g_n is given by the relation $x_n = g_n a_n$ for all $x = (x_n)_n$, is a probability space isomorphism that identifies α with $G \curvearrowright \varprojlim G/G_n$.

These profinite actions are very interesting when it comes to calculating fundamental groups, as it is easy to see elements in the commensurator of the action. Indeed, let $G \overset{\alpha}{\curvearrowright} \varprojlim G/G_n$ be a profinite action of a countable group, where $G_n < G$ are a descending chain of finite index subgroups of G. Then $G \overset{\alpha}{\curvearrowright} \varprojlim G/G_n$ is induced from $G_i \curvearrowright \varprojlim G_i/G_n$ for any i. In particular, whenever, $G_i \cong G_j$ such that the associated actions are conjugate, this gives us an element in the commensurator of α. In some cases, even more can be said, as is shown by the following theorem by Popa and Vaes.

Theorem 2.3. *[PV08c, Theorem 5.2] Let Γ be a group having a normal, non-virtually abelian subgroup Σ with the relative property (T) and with Γ/Σ being finitely generated. Let $\Gamma \supset \Gamma_1 \supset \Gamma_2 \supset \cdots$ be a decreasing sequence of finite index subgroups such that the action $\Gamma \curvearrowright (X, \mu) := \varprojlim \Gamma/\Gamma_n$ is essentially free. Consider the diagonal product action $\Gamma \curvearrowright X \times [0,1]^\Gamma$ of $\Gamma \curvearrowright X$ and the Bernoulli action $\Gamma \curvearrowright [0,1]^\Gamma$. Then the fundamental groups of the associated II$_1$ factor and II$_1$ equivalence relation are both equal to*

$$\Big\{\frac{[\Gamma : \Lambda_1]}{[\Gamma : \Lambda_2]} \quad \Big| \quad \Gamma_n \subset \Lambda_1 \cap \Lambda_2 \textit{ for large enough } n,$$

$$\Lambda_1 \curvearrowright \varprojlim \Lambda_1/\Gamma_n \textit{ conjugate to } \Lambda_2 \curvearrowright \varprojlim \Lambda_2/\Gamma_n\Big\}$$

This result was obtained using cocycle superrigidity techniques from [Io08] and [Po05]. Looking at the proof of this theorem, one notices that elements in the commensurator of the diagonal action only come from elements in the commensurator of the profinite action. In particular, if we are only interested in the fundamental group of the associated equivalence relation, the same result still holds when considering only $\Gamma \curvearrowright \varprojlim \Gamma/\Gamma_n$.

Applications of Lemma 2.1

Example 1

Initially we hoped that this lemma would allow us to find a second Cartan subalgebra such that the fundamental group of the associated equivalence relation was strictly smaller then the fundamental group of the original II_1 factor. A first example we considered was the following: let $Z = (\mathbb{Z} \oplus \mathbb{Z})^3$, $Z_0 = \{(z, z) \mid z \in \mathbb{Z}\}^3$ and $\Gamma = \mathrm{SL}_3(\mathbb{Z})$. Let $\mathcal{Z} := ((\mathbb{Z}_2^3 \times \mathbb{Z}_5^3) \times (\mathbb{Z}_5^3 \times \mathbb{Z}_3^3))$. We consider the action

$$Z \curvearrowright \mathcal{Z} : (z_1, z_2) \cdot (w, x, y, z) = (i_2(z_1) + w, i_5(z_2) + x, i_5(z_1) + y, i_3(z_2) + z)$$

where $i_p : \mathbb{Z}^3 \hookrightarrow \mathbb{Z}_p^3$ is the natural embedding. Finally Γ acts on both $\mathcal{Z}$ and Z by left multiplication. Note that

$$\mathcal{Z} = \lim_{\leftarrow} \frac{(\mathbb{Z} \oplus \mathbb{Z})^3 \rtimes \mathrm{SL}_3(\mathbb{Z})}{(10^n\mathbb{Z} \oplus 15^n\mathbb{Z})^3 \rtimes \mathrm{SL}_3(\mathbb{Z})},$$

so we are actually in the setting of a profinite action. Furthermore as $(\mathbb{Z} \oplus \mathbb{Z})^3 \rtimes \mathrm{SL}_3(\mathbb{Z})$ has property (T), we are in the setting of Theorem 2.3.

We check that this setting indeed satisfies the conditions of Lemma 2.1. Clearly $\mathcal{Z}$ is a compact abelian group and Z is a countable subgroup. Furthermore Z_0 is an infinite subgroup of Z and Γ preserves both Z and Z_0. Now as $2, 3$ and 5 are coprime, $\overline{Z_0} = \{((a, c), (c, b)) \mid a \in \mathbb{Z}_2^3, b \in \mathbb{Z}_3^3, c \in \mathbb{Z}_5^3\} =: \mathcal{Z}_0$. The other conditions that need to be satisfied are the following.

1. For all $g \in \Gamma$ the set $S_g := \{z - \alpha_g(z) \mid z \in Z_0\}$ is either trivial or infinite.
2. For all $g \in \Gamma_0 := \{g \in \Gamma \mid \alpha_g(z) = z \text{ for all } z \in Z_0\} \backslash \{e\}$ the space $\{x \in \frac{\mathcal{Z}}{\mathcal{Z}_0} \mid \alpha_g(x) = x\}$ has finite index in $\frac{\mathcal{Z}}{\mathcal{Z}_0}$.
3. $Z \cap \mathcal{Z}_0 = Z_0$.

Take $g \in \Gamma$. Suppose $\{z - \alpha_g(z) \mid z \in Z_0\}$ is not trivial. Take z such that $z - \alpha_g(z) \in S_g \backslash \{e\}$. Then any integer multiple kz of z will give us a non-trivial value of $kz - \alpha_g(kz)$, where we denote by k the diagonal matrix with k on the diagonal. It is clear that all these elements will be different, hence S_g is infinite, proving the first condition is satisfied.

For the second condition, note that $\Gamma_0 = \{e\}$. Indeed, no matrix in $\mathrm{SL}_3(\mathbb{Z})$ fixes all elements in $\mathbb{Z}^3$. So it remains to check the third condition.

Now $\mathcal{Z}_0 = \{((a, c), (c, b)) \mid a \in \mathbb{Z}_2^3, b \in \mathbb{Z}_3^3, c \in \mathbb{Z}_5^3\}$ as we remarked before, and hence $\mathcal{Z}_0 \cap Z$ are those elements (z_1, z_2) of Z for which $i_2(z_1) = i_5(z_1) = i_5(z_2) =$

$i_3(z_2)$. This is indeed Z_0, so this example satisfies the conditions of Lemma 2.1. So we know that $\mathrm{L}^\infty(\widehat{Z_0} \times \frac{\mathcal{Z}}{\mathcal{Z}_0})$ is a Cartan subalgebra of $\mathrm{L}^\infty(\mathcal{Z}) \rtimes (Z \rtimes \Gamma)$, and the induced equivalence relation is the orbit equivalence relation of the action $(\widehat{\mathcal{Z}_0} \times \frac{Z}{Z_0}) \rtimes \Gamma \curvearrowright \widehat{Z_0} \times \frac{\mathcal{Z}}{\mathcal{Z}_0}$. Denote by $\mathcal{R}_1$ the original equivalence relation, and by $\mathcal{R}_2$ the new equivalence relation, so the orbit equivalence relation of the action $(\widehat{\mathcal{Z}_0} \times \frac{Z}{Z_0}) \rtimes \Gamma \curvearrowright \widehat{Z_0} \times \frac{\mathcal{Z}}{\mathcal{Z}_0}$.

We first show, using Theorem 2.3, that $\mathcal{F}(\mathcal{R}_1) \supset \langle 2^3, 3^3, 5^3 \rangle$. It suffices to show that

$$(2\mathbb{Z} \oplus \mathbb{Z})^3 \rtimes \mathrm{SL}_3(\mathbb{Z}) \curvearrowright \left(\left(2\mathbb{Z}_2^3 \times \mathbb{Z}_5^3\right) \times \left(\mathbb{Z}_5^3 \times \mathbb{Z}_3^3\right)\right)$$

is conjugate to

$$(\mathbb{Z} \oplus \mathbb{Z})^3 \rtimes \mathrm{SL}_3(\mathbb{Z}) \curvearrowright ((\mathbb{Z}_2^3 \times \mathbb{Z}_5^3) \times (\mathbb{Z}_5^3 \times \mathbb{Z}_3^3)).$$

But this is immediate, by setting

$$\delta : (2\mathbb{Z} \oplus \mathbb{Z})^3 \rtimes \mathrm{SL}_3(\mathbb{Z}) \to (\mathbb{Z} \oplus \mathbb{Z})^3 \rtimes \mathrm{SL}_3(\mathbb{Z}) : (x, y, g) \mapsto (\frac{x}{2}, y, g)$$

and

$$\begin{aligned} \Delta : \quad & \left(\left(2\mathbb{Z}_2^3 \times \mathbb{Z}_5^3\right) \times \left(\mathbb{Z}_5^3 \times \mathbb{Z}_3^3\right)\right) \to ((\mathbb{Z}_2^3 \times \mathbb{Z}_5^3) \times (\mathbb{Z}_5^3 \times \mathbb{Z}_3^3)) : \\ & (w, x, y, z) \mapsto (\frac{w}{2}, x, y, z). \end{aligned}$$

Initially, we hoped that in constructing this new equivalence relation $\mathcal{R}_2$, we could ‘quotient out’ part of this fundamental group. However, $\widehat{\mathcal{Z}_0} \curvearrowright \widehat{Z_0}$ still produces these same elements in the fundamental group. Indeed, recall that the Prüfer p-group is the group $Z(p^\infty) = \{\exp(\frac{2\pi i m}{p^n}) \mid m, n \in \mathbb{Z}_+\}$ (or alternatively $Z(p^\infty) = \mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$). This group is the Pontryagin dual of the p-adic integers, and acts on the torus in a natural way. So we have

$$\widehat{\mathcal{Z}_0} = \left\{((a, c), (c, b)) \mid a \in Z(2^\infty)^3, b \in Z(3^\infty)^3, c \in Z(5^\infty)^3\right\} \curvearrowright \mathbb{T}^3.$$

Now $Z(p^\infty)$ has finite normal subgroups N_k with p^k elements for any $k \in \mathbb{N}$ such that $Z(p^\infty) \cong Z(p^\infty)/N_k$. It follows from Proposition 1.51 that this implies $\langle 2^3, 3^3, 5^3 \rangle \subset \mathcal{F}(\mathcal{R}_2)$.

From this example, it became clear to us that any interesting case built from the setting of Lemma 2.1 would involve a Cartan, twisted by a non-trivial cocycle. This made us look for examples where the fundamental group of the original II_1 factor would be smaller than the fundamental group of the equivalence relation associated to the second Cartan subalgebra. A first try in this setting was the next example.

Example 2

In this second example we considered, set $Z = (\mathbb{Z}[\frac{1}{2}])^3$, $Z_0 = \mathbb{Z}^3$ and $\mathcal{Z} = (\widehat{\mathbb{Z}[\frac{1}{2}]})^3$. We still set $\Gamma = \mathrm{SL}_3(\mathbb{Z})$. In this setting, view Z as a subgroup of $\mathcal{Z}$ by embedding it through $i : \mathbb{Z}[\frac{1}{2}] \hookrightarrow \widehat{\mathbb{Z}[\frac{1}{2}]} : x \mapsto \phi_x$ where

$$\phi_x : \mathbb{Z}[\frac{1}{2}] \to \mathbb{C} : y \mapsto \exp(2\pi xyi).$$

As before, we need to check that conditions 1,2 and 3 hold.

First take $g \in \Gamma$ and suppose $S_g = \{z - \alpha_g(z) \mid z \in Z_0\}$ is not trivial. Take z such that $z - \alpha_g(z) \in S_g \backslash \{e\}$. Then any integer multiple kz of z will give us a non-trivial value of $kz - \alpha_g(kz)$, where we denote by k the diagonal matrix with k on the diagonal. It is clear that all these elements will be different, hence S_g is infinite, proving the first condition is satisfied.

The second condition is also satisfied, as $\Gamma_0 = \{e\}$. Indeed, no matrix in $\mathrm{SL}_3(\mathbb{Z})$ fixes all elements in $\mathbb{Z}^3$. So it remains to check that $\overline{Z_0} \cap Z = Z_0$.

Note that we can view $\mathcal{Z}$ as an inverse limit of $(\mathbb{T})^3$ in the following sense: let $T_n = \mathbb{T}$ and set $p_n : T_{n+1} \to T_n : z \mapsto z^2$. Then $\mathcal{Z} = (\varprojlim T_n)^3$, as this is exactly dualising $\mathbb{Z}[\frac{1}{2}] = \varinjlim 2^{-n}\mathbb{Z}$. Whenever $\omega \in \widehat{\mathbb{Z}[\frac{1}{2}]}$ we associate to it the element $\big(\omega(\frac{1}{2^n})\big)_n \in \varprojlim T_n$. Seen like this, $\overline{Z_0}$ are those elements $(\omega_n)_n$ in $\mathcal{Z}$ such that $\omega_1 = 1$. In particular, $\overline{Z_0} = \mathbb{Z}_2^3$. Furthermore this implies that $\mathbb{Z}_2 \cap \mathbb{Z}[\frac{1}{2}] = \mathbb{Z}$, as $\phi_x(1) = 1$ assures x is an integer. So this example satisfies the conditions of Lemma 2.1. Now we can consider the equivalence relation associated to the second Cartan subalgebra, which is

$$\left(\widehat{\mathbb{Z}_2^3} \times \left(\frac{\mathbb{Z}[\frac{1}{2}]}{\mathbb{Z}}\right)^3\right) \rtimes \mathrm{SL}_3(\mathbb{Z}) \curvearrowright \widehat{\mathbb{Z}^3} \times \left(\widehat{\frac{\mathbb{Z}[\frac{1}{2}]}{\mathbb{Z}_2}}\right)^3.$$

Write (as before) $\mathcal{R}_1$ for the original equivalence relation, and $\mathcal{R}_2$ for the equivalence relation associated to the second Cartan subalgebra. In this setting, $\{8^n \mid n \in \mathbb{Z}\} \subset \mathcal{F}(\mathcal{R}_2)$. Indeed, as $\widehat{\mathbb{Z}_2} = Z(2^\infty)$ has finite normal subgroups N_k with 2^k elements for any $k \in \mathbb{N}$, such that $Z(2^\infty) \cong Z(2^\infty)/N_k$, this follows from Proposition 1.51.

There seem to be no immediate candidates for elements in the fundamental group of $\mathcal{R}_1$. However, we will show that $\mathcal{R}_1$ is orbit equivalent to an equivalence relation that does admit such ‘immediate candidates’. The idea is based on the following principle.

Proposition 2.4. *Let K_1, K_2 be compact abelian groups with their Haar measures, and $H_1 < K_1, H_2 < K_2$ countable dense subgroups. Let $\theta : K_1 \to K_2$ be a probability measure preserving isomorphism. If*

- $\theta(H_1 + x) = H_2 + \theta(x)$ *for almost all* $x \in K_1$,
- $\theta(x) + \theta(y) \in H_2 + \theta(x + y)$ *for almost all* $(x, y) \in K_1 \times K_1$ *and*
- $\theta^{-1}(x) + \theta^{-1}(y) \in H_1 + \theta^{-1}(x + y)$ *for almost all* $(x, y) \in K_2 \times K_2$,

then $H_1^3 \rtimes \mathrm{SL}_3(\mathbb{Z}) \curvearrowright K_1^3$ *and* $H_2^3 \rtimes \mathrm{SL}_3(\mathbb{Z}) \curvearrowright K_2^3$ *are orbit equivalent through* $\Phi =: \theta \times \theta \times \theta$.

Proof. Recall that $\mathrm{SL}_3(\mathbb{Z})$ is generated by matrices E_{ij} where E_{ij} is the unit matrix with an extra 1 on place (i, j). It suffices to check that for a generating matrix E_{ij} in $\mathrm{SL}_3(\mathbb{Z})$ we have that there is an element $(g, M) \in H_2^3 \rtimes \mathrm{SL}_3(\mathbb{Z})$ such that $\Phi(E_{ij} \cdot x) = (g, M) \cdot \Phi(x)$. Now

$$\Phi\left(\begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \cdot \begin{pmatrix}x\\ y\\ z\end{pmatrix}\right) = \left(\begin{pmatrix}\theta(x+y)\\ \theta(y)\\ \theta(z)\end{pmatrix}\right) = \begin{pmatrix}h + \theta(x) + \theta(y)\\ \theta(y)\\ \theta(z)\end{pmatrix}$$

$$= \begin{pmatrix}h\\ 0\\ 0\end{pmatrix} \begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \cdot \begin{pmatrix}\theta(x)\\ \theta(y)\\ \theta(z)\end{pmatrix}$$

□

We can use this to show that $\mathcal{R}_1$ is orbit equivalent to

$$\left(\frac{\mathbb{Z}[\frac{1}{2}]}{\mathbb{Z}} \times \mathbb{Z}\right)^3 \rtimes \mathrm{SL}_3(\mathbb{Z}) \curvearrowright \mathbb{T}^3 \times \mathbb{Z}_2^3.$$

Note that if this holds, $\{8^n \mid n \in \mathbb{Z}\} \subset \mathcal{F}(\mathcal{R}_1)$ as $\frac{\mathbb{Z}[\frac{1}{2}]}{\mathbb{Z}}$ is again the Prüfer 2-group. Let

$$\theta : \lim_{\leftarrow} T_n \to \mathbb{T} \times \mathbb{Z}_2 : (\exp(2\pi i y_n))_n \mapsto (\exp(2\pi i y_1), (2^n y_n - y_1)_n),$$

and the inverse map

$$\theta^{-1} : \mathbb{T} \times \mathbb{Z}_2 \to \lim_{\leftarrow} T_n : (\exp(2\pi i x), (y_n)_n) \mapsto (\exp(2\pi i \frac{x + y_n}{2^n}))_n.$$

By proposition 2.4 it suffices to show that

1. $\theta((x_n)_n)\theta((y_n)_n)(\theta((x_n+y_n)_n))^{-1} \in \frac{\mathbb{Z}[\frac{1}{2}]}{\mathbb{Z}} \times \mathbb{Z}$
2. $\theta^{-1}(\exp(2\pi i x),(y_n)_n) \cdot \theta^{-1}(\exp(2\pi i x'),(y'_n)_n)$
   $\cdot (\theta^{-1}(\exp(2\pi i(x+x')),(y_n+y'_n)_n))^{-1} \in \mathbb{Z}[\frac{1}{2}]$
3. $\theta(\mathbb{Z}[\frac{1}{2}] \cdot x) = \frac{\mathbb{Z}[\frac{1}{2}]}{\mathbb{Z}} \times \mathbb{Z} \cdot \theta(x)$

The third claim follows from 1 and 2, as $\theta(\mathbb{Z}[\frac{1}{2}]) = \frac{\mathbb{Z}[\frac{1}{2}]}{\mathbb{Z}} \times \mathbb{Z}$. Indeed, let $\frac{a}{2^k} \in \mathbb{Z}[\frac{1}{2}]$. Then the associated element in $\varprojlim T_n$ is $\left(\exp(2\pi i \frac{a}{2^{k+n}})\right)_n$. Now we have

$$\theta\left((\frac{a}{2^{k+n}})_n\right) = \left(\exp(2\pi i \frac{a}{2^k}),(0)_n\right).$$

Now

$$\begin{aligned}
&\theta^{-1}(\exp(2\pi i x),(y_n)_n) \cdot \theta^{-1}(\exp(2\pi i x'),(y'_n)_n) \cdot \\
&(\theta^{-1}(\exp(2\pi i(x+x')),(y_n+y'_n)_n))^{-1} \\
&\overset{x+x'=x'' \bmod \mathbb{Z}}{=} \left(\exp(2\pi i(\frac{x+y_n+x'+y'_n-(x''+y_n+y'_n)}{2^n}))\right)_n \\
&= \left(\exp(2\pi i(\frac{x+x'-x''}{2^n}))\right)_n \in \mathbb{Z}[\frac{1}{2}],
\end{aligned}$$

showing that 2 holds. It remains to check that 1 holds.

$$\begin{aligned}
&\theta((x_n)_n)\theta((y_n)_n)(\theta((x_n+y_n)_n))^{-1} \\
&\overset{x_n+y_n=x'_n \bmod \mathbb{Z}}{=} (\exp(2\pi i(x_1+y_1-(x'_1))), \\
&(2^n x_n - x_1 + 2^n y_n - y_1 - (2^n(x'_n)-x'_1))_n) \in \frac{\mathbb{Z}[\frac{1}{2}]}{\mathbb{Z}} \times \mathbb{Z}.
\end{aligned}$$

We have shown that $\mathcal{R}_1$ is indeed orbit equivalent to

$$\left(\frac{\mathbb{Z}[\frac{1}{2}]}{\mathbb{Z}} \times \mathbb{Z}\right)^3 \rtimes \mathrm{SL}_3(\mathbb{Z}) \curvearrowright \mathbb{T}^3 \times \mathbb{Z}_2^3.$$

Hence this example was no longer interesting, as there is no reason to assume a priori that in this case one can obtain different fundamental groups.

2.3 Setting for chapters 4 and 5

A final application of this lemma will be studied more in depth in chapters 4 and 5. We will construct an example satisfying the conditions of the lemma. We then show that the fundamental group of the given II_1 factor is trivial (chapter 4), but the fundamental group of the equivalence relation associated to the second Cartan subalgebra is non-trivial (chapter 5). The setting of this example is the following.

Notation 2.5. Let $n \geq 6$ even and define Σ as

$$\Sigma := \begin{pmatrix} \mathrm{SL}_2(\mathbb{Z}) & 0 & \dots & 0 \\ 0 & \mathrm{SL}_2(\mathbb{Z}) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathrm{SL}_2(\mathbb{Z}) \end{pmatrix} < \mathrm{SL}_n(\mathbb{Z}) \; .$$

Set

$$\Gamma = \mathrm{SL}_n(\mathbb{Z}) *_\Sigma (\Sigma \times \Lambda) \; ,$$

where Λ is a countable infinite group. Let $q : \Gamma \to \mathrm{SL}_n(\mathbb{Z})$ be the quotient map, i.e. $q(g) = g$ for all $g \in \mathrm{SL}_n(\mathbb{Z})$ and $q(\lambda) = e$ for all $\lambda \in \Lambda$. Fix a prime p and consider the action $\Gamma \overset{\alpha}{\curvearrowright} \mathbb{Z}_p^n$ through q. Remark that this action preserves $\mathbb{Z}^n$. Write $G := \mathbb{Z}^n \rtimes \Gamma$.

Define $X := (\mathbb{Z}_p^n)^\Gamma$. Embed $\mathbb{Z}_p^n$ into X by

$$i : \mathbb{Z}_p^n \to X : z \mapsto (\alpha_g(z))_{g \in \Gamma} .$$

Now consider the action $G \curvearrowright X$, where $\mathbb{Z}^n$ acts by translation after embedding by i, and Γ acts by Bernoulli shift. One can see that this is a free ergodic probability measure preserving action. We denote by M

$$M := \mathrm{L}^\infty(X) \rtimes G = \mathrm{L}^\infty((\mathbb{Z}_p^n)^\Gamma) \rtimes (\mathbb{Z}^n \rtimes (\mathrm{SL}_n(\mathbb{Z}) *_\Sigma (\Sigma \times \Lambda)))$$

the associated group measure space II_1 factor.

Originally, we looked at the less general and artificial construction

$$\mathbb{Z}^3 \rtimes \mathrm{SL}_3(\mathbb{Z}) \curvearrowright (\mathbb{Z}_2^3)^{\mathrm{SL}_3(\mathbb{Z})}.$$

However, to prove that the fundamental group of $\mathrm{L}^\infty((\mathbb{Z}_2^3)^{\mathrm{SL}_3(\mathbb{Z})}) \rtimes (\mathbb{Z}^3 \rtimes \mathrm{SL}_3(\mathbb{Z}))$ is trivial, we needed to show that partial automorphisms of this II_1 factor embed $\mathrm{L}^\infty((\mathbb{Z}_2^3)^{\mathrm{SL}_3(\mathbb{Z})}) \rtimes (\mathbb{Z}^3)$ into itself. To obtain this, we switched to an amalgamated free product setting, as many embedding results are known for these kinds of constructions. Furthermore to calculate the fundamental group of the second equivalence relation, we used a result by Popa and Vaes that had as a condition the fact that $n \geq 5$. Thus the above example was constructed. However in the original example, and through many of the same results, for $n \geq 5$ instead of $n = 3$, one can show through cocycle superrigidity arguments that the fundamental group of the original equivalence relation is trivial, whereas the equivalence relation associated to the second Cartan subalgebra has non-trivial fundamental group.

Chapter 3

A II_1 factor with trivial fundamental group . . .

The goal of this chapter is to prove the following theorem.

Theorem 3.1. *With $G \curvearrowright X$ and $M = \mathrm{L}^\infty(X) \rtimes G$ as in Notation 2.5, we have $\mathcal{F}(M) = \{1\}$. Hence also $\mathcal{F}(\mathcal{R}(G \curvearrowright X)) = \{1\}$.*

To prove this, we will need to describe the stable automorphisms of M. Indeed, whenever $\alpha : M \to M^t$ is an isomorphism between M and an amplification of M, we will show that $t = 1$. From here on out let us assume $t \leq 1$.

We first describe stable automorphisms on a part of M (§3.1). Then we will use cocycle superrigidity techniques (§3.2) to conclude that theorem 3.1 holds (§3.3).

3.1 A partial description of stable automorphisms of M

Definition 3.2. A center-valued trace on a von Neumann algebra M is a conditional expectation $E_{\mathcal{Z}(M)}$ from M onto its center $\mathcal{Z}(M)$ such that $E_{\mathcal{Z}(M)}(xy) = E_{\mathcal{Z}(M)}(yx)$ for all $x, y \in M$.

One can show ([Ta02, Theorem 2.6]) that any finite von Neumann algebra carries a unique normal faithful center-valued trace.

Definition 3.3. Let p be a projection in a von Neumann algebra M. Denote by $\mathcal{Z}(M)$ the center of M and by $z_M(p)$ the central support of p in M. We define the $*$-isomorphism

$$\varphi_p : \mathcal{Z}(M)z_M(p) \to \mathcal{Z}(pMp) \quad \text{by} \;\; \varphi_p(m) = mp \, .$$

Remark that $\mathcal{Z}(pMp) = \mathcal{Z}(M)p$. A proof for this can be found in [Di69, Proposition 1]. Note that φ_p is indeed a $*$-isomorphism: φ_p is clearly a $*$-homomorphism. To see that φ_p is surjective, take $y \in \mathcal{Z}(M)p$. Then $y = mp$ for some $m \in \mathcal{Z}(M)$. Set $x := mz_M(p)$ then $\varphi_p(x) = y$. Finally, assume $m \in \mathcal{Z}(M)z_M(p)$ and $\varphi_p(m) = 0$. Then $E_{\mathcal{Z}(M)}(mp) = 0$ and hence $mE_{\mathcal{Z}(M)}(p)z_M(p) = 0$. As $z_M(p)$ is the right support of $E_{\mathcal{Z}(M)}(p)$ we have $mE_{\mathcal{Z}(M)}(p)z_M(p) = 0$ or $E_{\mathcal{Z}(M)}(p)z_M(p)m = 0$. But then $z_M(p)m = 0$.

To lighten notation, we will write $K = \mathbb{Z}_p^n$, $Z = \mathbb{Z}^n$, $B = \mathrm{L}^\infty((\mathbb{Z}_p^n)^\Gamma) \rtimes \mathbb{Z}^n$, $M_1 = B \rtimes \mathrm{SL}_n(\mathbb{Z})$ and $M_2 = (B \rtimes \Sigma)\overline{\otimes}\mathcal{L}(\Lambda)$. Note that with this notation, $M = M_1 \underset{B \rtimes \Sigma}{*} M_2$. Remark also that $\mathcal{Z}(B) = \mathrm{L}^\infty(K^\Gamma / i(K))$.

The goal of this section is to prove two lemmas, which describe stable automorphisms of M on a part of M. The first lemma shows that if $\alpha : M \to M^t$ is an isomorphism of M to some amplification of M, after cutting with projections in B, we get that a corner of B is mapped to another corner of B by α. The second lemma then shows that on this part of B, α can be seen as a stable orbit equivalence of $\Gamma \curvearrowright \frac{K^\Gamma}{i(K)}$.

Lemma 3.4. *Let $\alpha : M \to pMp$ be a stable automorphism of M as before. There exist projections $q, r \in B$ such that after a unitary conjugacy of α, we have $\alpha(q) = r$ and $\alpha(qBq) = rBr$.*

Proof. Note that B is amenable and that $M = B \rtimes \Gamma$. By [Io12, Theorem 7.1] we get that $\alpha(B) \prec_M B$ and that $B \prec_M \alpha(B)$. Remark that $B' \cap M = \mathrm{L}^\infty(X/i(K)) = \mathcal{Z}(B)$. [Va07, Lemma 3.5] then gives

$$\mathcal{Z}(B) \prec_M \alpha(\mathcal{Z}(B)) \quad \text{and} \quad \alpha(\mathcal{Z}(B)) \prec_M \mathcal{Z}(B) \, . \tag{3.1}$$

Since $\mathcal{Z}(B)$ is regular in M and $\alpha(\mathcal{Z}(B))$ is regular in pMp, (3.1) implies that $\mathrm{L}^2(M)p$ can be written as a direct sum of $\mathcal{Z}(B)$-$\alpha(\mathcal{Z}(B))$-bimodules with $\dim(-_{\alpha(\mathcal{Z}(B))})$ finite and that $p\mathrm{L}^2(M)$ can be written as a direct sum of $\alpha(\mathcal{Z}(B))$-$\mathcal{Z}(B)$-bimodules with $\dim(-_{\mathcal{Z}(B)})$ finite.

So $\mathrm{L}^2(M)p$ admits a non-zero $\mathcal{Z}(B)$-$\alpha(\mathcal{Z}(B))$-subbimodule with $\dim({}_{\mathcal{Z}(B)}-)$ finite and $\dim(-_{\alpha(\mathcal{Z}(B))})$ finite. This means that there exists

- a projection $p \in M_n(\mathbb{C}) \otimes \alpha(\mathcal{Z}(B))$

- a non-zero partial isometry $v \in (M_{1,n}(\mathbb{C}) \otimes M)p$
- a unital $*$-homomorphism $\theta : \mathcal{Z}(B) \to p(M_n(\mathbb{C}) \otimes \alpha(\mathcal{Z}(B)))p$

such that $\theta(\mathcal{Z}(B)) \subset p(M_n(\mathbb{C}) \otimes \alpha(\mathcal{Z}(B)))p$ has finite index and $bv = v\theta(b)$ for all $b \in \mathcal{Z}(B)$. We will manipulate these to obtain a 'nice' $*$-isomorphism between corners of $\mathcal{Z}(B)$.

As $\theta(\mathcal{Z}(B)) \subset p(\mathrm{M}_n(\mathbb{C}) \otimes \alpha(\mathcal{Z}(B)))p$ is abelian and $\mathrm{M}_n(\mathbb{C}) \otimes \alpha(\mathcal{Z}(B))$ is a finite type I von Neumann algebra, there is a unitary $u \in \mathrm{M}_n(\mathbb{C}) \otimes \alpha(\mathcal{Z}(B))$ such that $u\theta(\mathcal{Z}(B))u^* \subset \mathrm{D}_n(\mathbb{C} \otimes \alpha(\mathcal{Z}(b))$. Set $\tilde{p} := upu^* \in \mathrm{D}_n(\mathbb{C}) \otimes \alpha(\mathcal{Z}(B))$. Then

$$u\theta(\mathcal{Z}(B))u^* \subset \tilde{p}(\mathrm{D}_n(\mathbb{C}) \otimes \alpha(\mathcal{Z}(B)))\tilde{p}.$$

Now $\theta(\mathcal{Z}(B)) \subset p(\mathrm{M}_n(\mathbb{C}) \otimes \alpha(\mathcal{Z}(B)))p$ has finite index, hence so does $u\theta(\mathcal{Z}(B))u^* \subset \tilde{p}(\mathrm{M}_n(\mathbb{C}) \otimes \alpha(\mathcal{Z}(B)))\tilde{p}$. In particular, $u\theta(\mathcal{Z}(B))u^* \subset \tilde{p}(\mathrm{D}_n(\mathbb{C}) \otimes \alpha(\mathcal{Z}(B)))\tilde{p}$ has finite index. So we replace θ by $u\theta(\cdot)u^* : \mathcal{Z}(B) \to (\mathrm{D}_n(\mathbb{C}) \otimes \alpha(\mathcal{Z}(B)))p$, p by $\tilde{p}$ and v by vu^*. We stil have $bv = v\theta(b)$ for all $b \in \mathcal{Z}(B)$.

Now choose $1 \leq i \leq n$ such that $ve_{ii} \neq 0$, then $pe_{ii} \neq 0$ since $v = vp$. Note that

$$(\mathrm{D}_n(\mathbb{C}) \otimes \alpha(\mathcal{Z}(B)))pe_{ii} \cong \alpha(\mathcal{Z}(B))p_{ii},$$

where p_{ii} is the i^{th} element on the diagonal of p. Replacing θ by $\theta(\cdot)e_{ii}$, p by pe_{ii} and v by the partial isometry obtained from the polar decomposition of ve_{ii}, we have found a projection $p \in \alpha(\mathcal{Z}(B))$, a non-zero partial isometry $v \in Mp$ and a unital $*$-homomorphism $\theta : \mathcal{Z}(B) \to \alpha(\mathcal{Z}(B))p$ such that $\theta(\mathcal{Z}(B)) \subset \alpha(\mathcal{Z}(B))p$ has finite index and $bv = v\theta(b)$ for all $b \in \mathcal{Z}(B)$.

Choose a non-zero projection $q \in \mathcal{Z}(B)$ such that $\theta_{|\mathcal{Z}(B)}$ is injective and $\theta(\mathcal{Z}(B)q) = \theta(\mathcal{Z}(B))$. Then $\theta(\mathcal{Z}(B)q) \subset \alpha(\mathcal{Z}(B))p$ has finite index, so there is a non-zero projection $t \in \theta(\mathcal{Z}(B)q)$ such that $\theta(\mathcal{Z}(B)q)t = \alpha(\mathcal{Z}(B))pt$. Now $t \leq p$ so $\alpha(\mathcal{Z}(B))pt = \alpha(\mathcal{Z}(B))t$. Let $z \in \mathcal{Z}(B)q$ be the unique projection such that $t = \theta(z)$, then $\theta : \mathcal{Z}(B)z \to \alpha(\mathcal{Z}(B))t$ is a unital $*$-isomorphism such that $bv = v\theta(b)$ for all $b \in \mathcal{Z}(B)z$.

So we found

- projections $z \in \mathcal{Z}(B), t \in \alpha(\mathcal{Z}(B))$
- a non-zero partial isometry $v \in Mt$
- a unital $*$-isomorphism $\theta : \mathcal{Z}(B)z \to \alpha(\mathcal{Z}(B))t$

such that $bv = v\theta(b)$ for all $b \in \mathcal{Z}(B)z$.

Remark that $\mathcal{Z}(B)' \cap M = B, vv^* \in Bz$ and $v^*v \in \alpha(B)t$. In particular $vv^* \leq s$ and $v^*v \leq t$. We claim that $v\alpha(\mathcal{Z}(B))v^* = vv^*\mathcal{Z}(B)vv^*$.

$\boxed{\subset}$ Let $b \in \alpha(\mathcal{Z}(B))$ and take $c \in \mathcal{Z}(B)$ such that $bt = \theta(cs)$. Then $vbv^* = vbtv^* = v\theta(cs)v^* = csvv^* = cvv^* = vv^*cvv^*$ as $vv^* \in B$.

$\boxed{\supset}$ Whenever $b \in \mathcal{Z}(B)$ we have $vv^*bvv^* = vv^*bsvv^* = vv^*v\theta(bs)v^* = v\theta(bs)v^* \in v\alpha(\mathcal{Z}(B))v^*$.

But then $v\alpha(B)v^* = vv^*Bvv^*$. So we extend v to a unitary $u \in M$ and define $q := \alpha^{-1}(v^*v)$ and $r := vv^*$. This gives us

$$(\mathrm{Ad}\, u \circ \alpha)q = r \text{ and } (\mathrm{Ad}\, u \circ \alpha)(qBq) = rBr,$$

which proves the lemma. $\square$

For any projection $q \in B$ define the set $\mathcal{I}_q$ as follows:

$$\begin{aligned}\mathcal{I}_q := \{v \in qMq \quad | \quad & v \text{ is a partial isometry with } v^*v, vv^* \in qBq \\ & \text{and } vBv^* = vv^*Bvv^*\} \ .\end{aligned}$$

Note that for every $v \in \mathcal{I}_q$ the map $\mathrm{Ad}\, v$ is a $*$-isomorphism between v^*vBv^*v and vv^*Bvv^*. For every $v \in \mathcal{I}_q$ there exists a unique $*$-isomorphism $\theta_v : \mathcal{Z}(B)z_B(v^*v) \to \mathcal{Z}(B)z_B(vv^*)$ satisfying

$$\theta_v(b)v = vb \quad \text{for all} \quad b \in \mathcal{Z}(B)z_B(v^*v) \ .$$

Indeed, one checks that $\varphi_{vv^*}^{-1} \circ \mathrm{Ad}\, v \circ \varphi_{v^*v}$ satisfies the conditions and is unique as such.

Furthermore, whenever $p \in B$ is a projection, we will denote by $U_p \subset \frac{K^\Gamma}{i(K)}$ the set such that $z_B(p) = \chi_{U_p}$.

To prove Lemma 3.6, we first prove the following lemma.

Lemma 3.5. *In the same setting as before, we have for every projection $p \in B$ that*

$$\bigcup_{v \in \mathcal{I}_p} \mathrm{graph}\, \theta_v = \mathcal{R}(\Gamma \curvearrowright K^\Gamma / i(K))_{|U_p}.$$

Proof. We prove both inclusions.

$\boxed{\subset}$ Let $p \in B$ and $v \in \mathcal{I}_p$. Write $v = \sum_{g \in \Gamma} b_g u_g$ with $b_g \in B$. For all $b \in \mathcal{Z}(B)z_B(v^*v)$ we have $\theta_v(b)v = vb$ and hence

$$\sum_{g \in \Gamma} \theta_v(b) b_g u_g = \sum_{g \in \Gamma} b_g u_g b = \sum_{g \in \Gamma} b(g^{-1} \cdot\ ) b_g u_g.$$

So for all $g \in \Gamma$ and $b \in \mathcal{Z}(B)z_B(vv^*)$ we have $\theta_v(b)b_g = b(g^{-1}\cdot\,)b_g$. This implies that $z_B(b_g)\theta_v(b) = z_B(b_g)b(g^{-1}\cdot\,)$. As $z_B(vv^*) \leq \vee_{g\in\Gamma} z_B(b_g)$, we have proven $\boxed{\subset}$.

$\boxed{\supset}$ Fix $g \in \Gamma$ and denote by σ_g the Borel isomorphism of $\mathcal{Z}(B)$ given by the action of g. We prove there is a $v \in \mathcal{I}_p$ such that

$$\operatorname{graph}\theta_v = (\operatorname{graph}\sigma_g)_{|U_p}$$

where

$$(\operatorname{graph}\sigma_g)_{|U_p} := \{(b, \sigma_g(b)) \mid b \in \mathcal{Z}(B)z_B(p)\sigma_{g^{-1}}(z_B(p))\},$$

and we assume the latter is not empty.

Denote by $\mathcal{J} := \{v \in \mathcal{I}_p \mid \operatorname{graph}\theta_v \subset \operatorname{graph}\sigma_g\}$. We order this set partially by setting $u \leq v$ whenever $u^*u \leq v^*v$ and $vu^*u = u$. Note that $\mathcal{J}$ is not empty (as $0 \in \mathcal{J}$). So by Zorn's lemma, we can choose a maximal element v of $\mathcal{J}$. We claim that $z_B(v^*v) = z_B(p)\sigma_{g^{-1}}(z_B(p))$, i.e. v is the required partial isometry.

As $z_B(v^*v) = \theta_v(z_B(v^*v)) = \sigma_g(z_B(v^*v)) \leq z_B(p)$, we have that $z_B(v^*v) \leq z_B(p)\sigma_{g^{-1}}(z_B(p))$. Assume $z_B(v^*v) \neq z_B(p)\sigma_{g^{-1}}(z_B(p))$. We claim that there is a partial isometry $w \in \mathcal{I}_p$ such that $\operatorname{graph}\theta_w \subset \operatorname{graph}\sigma_g$ and $v < v + w$.

Note that $z_B(p)$ and $\sigma_{g^{-1}}(z_B(p)) - z_B(v^*v)$ are not orthogonal. Applying [Ta02, Lemma 1.7 p.292] we get a partial isometry $v_1 \in B$ such that $v_1v_1^* \leq p$ and $v_1^*v_1 \leq \sigma_{g^{-1}}(z_B(p)) - z_B(v^*v)$. As $z_B(\sigma_g(v_1^*v_1))$ and $z_B(p)$ are not orthogonal, applying [Ta02, Lemma 1.7 p.292] once more yields a partial isometry $v_2 \in B$ such that $v_2v_2^* \leq \sigma_g(v_1^*v_1)$ and $v_2^*v_2 \leq p$. We may replace v_1 in such a way that $v_2v_2^* = \sigma_g(v_1^*v_1)$. Define $w := v_2^*u_gv_1^* \in pMp$. Then $w^*w = v_1v_1^* \in pBp$ and $ww^* = v_2^*v_2 \in pBp$. As before $wBw^* = ww^*Bww^*$, so $w \in \mathcal{I}_p$. One checks that $\operatorname{graph}\theta_w \subset \operatorname{graph}\sigma_g$.

Furthermore $v + w$ is a partial isometry in $\mathcal{I}_p$ and $\operatorname{graph}\theta_{v+w} \subset \operatorname{graph}\sigma_g$. But that contradicts the maximality of v, proving $\boxed{\supset}$.

□

Lemma 3.6. *Let $\alpha : M \to pMp$ be a stable automorphism of M as before. There exist projections $q, r \in B$ such that the $*$-isomorphism $\Psi : \mathcal{Z}(B)z_B(q) \to \mathcal{Z}(B)z_B(r)$ given by the composition of*

$$\mathcal{Z}(B)z_B(q) \overset{\varphi_q}{\to} \mathcal{Z}(qBq) \overset{\alpha}{\to} \mathcal{Z}(rBr) \overset{\varphi_r^{-1}}{\to} \mathcal{Z}(B)z_B(r)$$

is a stable orbit equivalence of the action $\Gamma \curvearrowright K^\Gamma/i(K)$.

Proof. By Lemma 3.4, we find projections $q, r \in B$ such that $\alpha(q) = r$ and $\alpha(qBq) = rBr$. Denote by $\mathcal{Q} \subset X/i(K)$ the support of the projection $z_B(q)$.

By Lemma 3.5, the restricted orbit equivalence relation $\mathcal{R}(\Gamma \curvearrowright X/i(K))_{|\mathcal{Q}}$ is generated by the graphs of θ_v with $v \in \mathcal{I}_q$.

To conclude the proof it suffices to remark that $\alpha(\mathcal{I}_q) = \mathcal{I}_r$. This can easily be shown to follow from $vBv^* = vv^*Bvv^*$. □

3.2 Cocycle superrigidity techniques

Recall from [Po05] that an infinite subgroup H of a group Γ is wq-normal in Γ if there exists an increasing sequence $(H_n)_n$ of subgroups of Γ with $H_0 = H, \cup_n H_n = \Gamma$ and such that for all n the group H_n is generated by the elements $g \in \Gamma$ with $|gH_{n-1}g^{-1} \cap H_{n-1}| = \infty$.

Recall that an inclusion of groups $H \subset \Gamma$ has the relative property (T) of Kazhdan-Margulis if any unitary representation of Γ that almost contains the trivial representation of Γ must contain the trivial representation of H. We call such H a rigid subgroup of Γ.

The following lemma is a corollary of [Po05, Theorem 0.1].

Lemma 3.7. *Let K be a compact group and let Γ be a countable group. Assume that Γ admits an infinite rigid subgroup that is wq-normal in Γ. Assume that Γ acts on K by continuous group automorphisms $(\alpha_g)_{g\in\Gamma}$. Embed K in K^Γ by $i : K \to K^\Gamma : k \mapsto (\alpha_g(k))_{g\in\Gamma}$. Then the action $K \rtimes \Gamma \curvearrowright K^\Gamma$ where K acts by translation after embedding by i and Γ acts by Bernoulli shift, is $\mathcal{U}_{fin}$-cocycle superrigid (see definition 1.59).*

Proof. Let $\omega : (K \rtimes \Gamma) \times K^\Gamma \to \mathcal{G}$ be a 1-cocycle for the action $K \rtimes \Gamma \curvearrowright K^\Gamma$ with values in a Polish group of finite type. By Popa's cocycle superrigidity theorem [Po05, Theorem 0.1] we may assume that the restriction $\omega : \Gamma \times K^\Gamma \to \mathcal{G}$ is a group morphism δ, i.e. $\omega(g, x) = \delta(g)$ for all $g \in \Gamma$ and a.e. $x \in K^\Gamma$. It remains to prove that $\omega_{|K\times K^\Gamma}$ is a group morphism.

For all $g \in \Gamma, k \in K$ and a.e. $x \in K^\Gamma$ we have

$$\begin{aligned} \omega(\alpha_g(k), g \cdot x)\delta(g) &= \omega(\alpha_g(k), g \cdot x)\omega(g, x) \\ &= \omega(\alpha_g(k)g, x) \\ &= \omega(gk, x) \\ &= \omega(g, k \cdot x)\omega(k, x) \\ &= \delta(g)\omega(k, x) \end{aligned}$$

so that $\omega(\alpha_g(k), g \cdot x) = \delta(g)\omega(k, x)\delta(g)^{-1}$. Applying [PV08b, Lemma 5.4] to the restriction $\omega : K \times K^\Gamma \to \mathcal{G}$ then gives that $\omega_{|K \times K^\Gamma}$ is essentially independent of K^Γ. It follows that ω is a group morphism. □

The following theorem is a twisted version of Popa's cocycle superrigidity theorem [Po05, Theorem 5.5] for Bernoulli actions of groups admitting an infinite rigid subgroup that is wq-normal (see the discussion preceding Lemma 3.7 for the terminology). The theorem is an adapted version of [SV11, Theorem 11] for generalized 1-cocycles. A detailed proof of the case where you have a direct product of groups in stead of a semidirect product can be found in §4.4 of An Speelman's PhD Thesis 'Type II_1 factors with uncountably many non-conjugate Cartan subalgebras' ([Sp13]).

Definition 3.8. Let $(\sigma_g)_{g \in \Gamma}$ be an action of a countable group Γ on a von Neumann algebra N. Let $q \in N$ be a projection. A generalized 1-cocycle for the action $(\sigma_g)_{g \in \Gamma}$ on N with support q is a family of partial isometries $(\gamma_g)_{g \in \Gamma}$ in $qN\sigma_g(q)$, satisfying $\gamma_g \gamma_g^* = q, \gamma_g^* \gamma_g = \sigma_g(q)$ and $\gamma_{gh} = \gamma_g \, \sigma_g(\gamma_h)$ for all $g, h \in \Gamma$.

For any von Neumann algebra M we denote by M^∞ the von Neumann algebra $M^\infty := \mathrm{B}(\ell^2(\mathbb{N})) \mathbin{\overline{\otimes}} M$.

Theorem 3.9. *Let K be a compact group with countable subgroup $Z < K$. Let Γ be a countable group. Assume that Γ admits an infinite rigid subgroup that is wq-normal in Γ. Put $X = K^\Gamma$. Assume that Γ acts on K by continuous group automorphisms $(\alpha_g)_{g \in \Gamma}$ preserving Z. Embed Z in X by $i : Z \to X : z \mapsto (\alpha_g(z))_{g \in \Gamma}$. Put*

$$N := \mathrm{L}^\infty(X) \rtimes Z$$

where Z acts on X by translation after embedding by i. Denote by $(\sigma_g)_{g \in \Gamma}$ the action of Γ on N such that σ_g is the Bernoulli shift on $\mathrm{L}^\infty(X)$ and is given by α_g on $\mathcal{L}Z$. Let $p \in N$ be a non-zero projection.

- *Assume that $q \in (\mathcal{L}Z)^\infty$ is a projection and that $(\gamma_g)_{g\in\Gamma}$ is a generalized 1-cocycle for the action of Γ on $(\mathcal{L}Z)^\infty$ with support q. Assume that $v \in \mathrm{B}(\mathbb{C}, \ell^2(\mathbb{N})) \overline{\otimes} N$ is a partial isometry satisfying $v^*v = p$ and $vv^* = q$. We view $\mathrm{B}(\mathbb{C}, \ell^2(\mathbb{N})) \subset \mathrm{B}(\mathbb{C} \oplus \ell^2(\mathbb{N}), \mathbb{C} \oplus \ell^2(\mathbb{N}))$. Then the formula*

$$\omega_g := v^* \, \gamma_g \, \sigma_g(v) \tag{3.2}$$

defines a generalized 1-cocycle $(\omega_g)_{g\in\Gamma}$ for the action $(\sigma_g)_{g\in\Gamma}$ on N with support p.

- *Conversely, every generalized 1-cocycle for the action $(\sigma_g)_{g\in\Gamma}$ on N is of the above form with γ being uniquely determined in the following sense: if $(\gamma_g)_g$ and $(\varphi_g)_g$ both satisfy (3.2), then there is a unitary $u \in (\mathcal{L}Z)^\infty$ such that $\varphi_g = u\gamma_g\sigma_g(u^*)$ for all $g \in \Gamma$.*

Proof. It is clear that the formula in the theorem define generalized 1-cocycles. Conversely, let $p \in N$ be a projection and assume that the partial isometries $(\omega_g)_{g\in\Gamma}$ define a generalized 1-cocycle for the action $(\sigma_g)_{g\in\Gamma}$ on N with support p. Denote by $E_{\mathcal{Z}(N)} : \mathrm{L}^\infty(X) \rtimes Z \to \mathrm{L}^\infty(X/\overline{i(Z)})$ the trace preserving conditional expectation of N onto its center. Since p and $\sigma_g(p)$ are equivalent in N, we have

$$E_{\mathcal{Z}(N)}(p) = E_{\mathcal{Z}(N)}(\sigma_g(p)) = \sigma_g(E_{\mathcal{Z}(N)}(p)) \text{ for all } g \in \Gamma.$$

By ergodicity of $\Gamma \curvearrowright X/\overline{i(Z)}$, it follows that $E_{\mathcal{Z}(N)}(p)$ is constant, i.e. $E_{\mathcal{Z}(N)}(p) = \tau(p)$ where τ denotes the tracial state on N. But then p is equivalent in N to any projection in $\mathcal{L}Z$ of trace $\tau(p)$. So we may assume that $p \in \mathcal{L}Z$.

Consider the action $Z \curvearrowright X \times K$ given by

$$z \cdot ((x_g)_g, k) = ((\alpha_g(z)x_g)_g, zk) \quad \text{for all } \; z \in Z, (x_g)_g \in X, k \in K \; .$$

Put $\mathcal{N} := \mathrm{L}^\infty(X \times K) \rtimes Z$. We embed $N \subset \mathcal{N}$ by identifying the element $Fu_z \in N$ with the element $(F \otimes 1)u_z \in \mathcal{N}$ whenever $F \in \mathrm{L}^\infty(X), z \in Z$. Also $(\sigma_g)_{g\in\Gamma}$ extends naturally to a group of automorphisms of $\mathcal{N}$ with $\sigma_g(1 \otimes F) = 1 \otimes (F \circ \alpha_{g^{-1}})$ for all $F \in \mathrm{L}^\infty(K)$. Define $P = \mathrm{L}^\infty(K) \rtimes Z$. View P as a subalgebra of $\mathcal{N}$ by identifying the element $Fu_z \in P$ with the element $(1 \otimes F)u_z \in \mathcal{N}$ whenever $F \in \mathrm{L}^\infty(K), z \in Z$. Remark that the restriction of σ_g to P is given by

$$\sigma_g(Fu_z) = (F \circ \alpha_{g^{-1}})u_{\alpha_g(z)} \text{ for all } F \in \mathrm{L}^\infty(K), z \in Z.$$

Denote by R the hyperfinite II_1 factor. Let $r \in R$ be a projection of trace $\tau(p)$. Recall that $p \in \mathcal{L}Z$. Denoting by $E_{\mathcal{Z}(P\overline{\otimes}R)} : P \overline{\otimes} R \to \mathrm{L}^\infty(K/\overline{i(Z)}) \otimes 1$ the trace preserving conditional expectation of $P \mathbin{\overline{\otimes}} R$ onto its center, we have that $E_{\mathcal{Z}(P\overline{\otimes}R)}(1 \otimes r) = E_{\mathcal{Z}(P\overline{\otimes}R)}(p \otimes 1) = \tau(p)$. It follows that $1 \otimes r$ and $p \otimes 1$ are equivalent in $P \mathbin{\overline{\otimes}} R$. Take $u \in P \mathbin{\overline{\otimes}} R$ with $uu^* = 1 \otimes r$ and $u^*u = p \otimes 1$. Now view u and ω_g in $\mathcal{N} \mathbin{\overline{\otimes}} R$ and define

$$\nu_g := u\omega_g(\sigma_g \otimes \mathrm{id})(u^*) \in \mathcal{U}(\mathcal{N} \mathbin{\overline{\otimes}} rRr) .$$

One verifies that ν_g is a 1-cocycle for the action $(\sigma_g \otimes \mathrm{id})_{g\in\Gamma}$ on $\mathcal{N} \mathbin{\overline{\otimes}} rRr$, i.e. a family of unitaries satisfying $\nu_{gh} = \nu_g\,(\sigma_g \otimes \mathrm{id})(\nu_h)$ for all $g, h \in \Gamma$.

Define $\Delta : X \times K \to X \times K$ by $\Delta((x_g)_g, k) = ((\alpha_g(k)x_g)_g, k)$ and denote by Δ_* the automorphism of $\mathrm{L}^\infty(X \times K)$ given by $\Delta_*(F) = F \circ \Delta^{-1}$. One checks that the formula

$$\Phi : \mathrm{L}^\infty(X) \mathbin{\overline{\otimes}} P \to \mathcal{N} : \Phi(F \otimes Gu_z) = \Delta_*(F \otimes G)u_z$$

for all $F \in \mathrm{L}^\infty(X), G \in \mathrm{L}^\infty(K)$ and $z \in Z$, defines a $*$-isomorphism satisfying $\Phi \circ (\sigma_g \otimes \sigma_g) = \sigma_g \circ \Phi$ for all $g \in \Gamma$. Define

$$\Psi = \Phi \otimes \mathrm{id}_R : \mathrm{L}^\infty(X) \mathbin{\overline{\otimes}} P \mathbin{\overline{\otimes}} R \to \mathcal{N} \mathbin{\overline{\otimes}} R .$$

Put $\mu_g := \Psi^{-1}(\nu_g)$. It follows that $(\mu_g)_{g\in\Gamma}$ is a 1-cocycle for the action $(\sigma_g \otimes \sigma_g \otimes \mathrm{id})_{g\in\Gamma}$ on $\mathrm{L}^\infty(X) \mathbin{\overline{\otimes}} P \mathbin{\overline{\otimes}} rRr$. Now $\Gamma \curvearrowright \mathrm{L}^\infty(X)$ is a Bernoulli action, and hence it is s-malleable and mixing. By assumption Γ admits an infinite rigid subgroup that is wq-normal and still acts weakly mixingly (as it is a Bernoulli action). So we can apply Popa's cocycle superrigidity theorem [Po05, Theorem 5.5]. Directly applying Ψ again, we find a unitary $v \in \mathcal{U}(\mathcal{N} \mathbin{\overline{\otimes}} rRr)$ and a 1-cocycle $\delta_g \in \mathcal{U}(P \mathbin{\overline{\otimes}} rRr)$ for the action $(\sigma_g \otimes \mathrm{id})_{g\in\Gamma}$ on $P \mathbin{\overline{\otimes}} rRr$ such that $\nu_g = v^*\delta_g(\sigma_g \otimes \mathrm{id})(v)$ for all $g \in \Gamma$.

Define $w := u^*vu$ and $\rho_g := u^*\delta_g(\sigma_g \otimes \mathrm{id})(u)$ for all $g \in \Gamma$. Then $w \in \mathcal{U}(p\mathcal{N}p \mathbin{\overline{\otimes}} R)$. Furthermore $\rho_g \in pP\sigma_g(p) \mathbin{\overline{\otimes}} R$ is a family of partial isometries satisfying $\rho_g\rho_g^* = p \otimes 1, \rho_g^*\rho_g = \sigma_g(p) \otimes 1$ and $\rho_{gh} = \rho_g(\sigma_g \otimes \mathrm{id})(\rho_h)$ such that

$$\omega_g = w^*\,\rho_g\,(\sigma_g \otimes \mathrm{id})(w) \quad \text{for all } \; g \in \Gamma .$$

Consider the basic construction for the inclusion $N \subset \mathcal{N} \mathbin{\overline{\otimes}} R$ denoted by $\mathcal{N}_1 := \langle \mathcal{N} \mathbin{\overline{\otimes}} R, e_N \rangle$. Put $T := we_Nw^*$. Since $\sigma_g(w) = \rho_g^*\,w\,\omega_g$, it follows that $\sigma_g(T) = \rho_g^*\,T\,\rho_g$. Also note that $T \in \mathrm{L}^2(\mathcal{N}_1)$, that $T = pT$ and that $\mathrm{Tr}(T) = \tau(p)$.

Define for every finite subset $\mathcal{F} \subset \Gamma$, the von Neumann subalgebras $N_\mathcal{F} \subset N$ and $\mathcal{N}_\mathcal{F} \subset \mathcal{N}$ given by

$$N_\mathcal{F} := \mathrm{L}^\infty(K^\mathcal{F}) \rtimes Z \;\text{ and }\; \mathcal{N}_\mathcal{F} := \mathrm{L}^\infty(K^\mathcal{F} \times K) \rtimes Z .$$

For every finite subset $\mathcal{F} \subset \Gamma$, we have the following commuting square

$$\begin{array}{ccc} N & \subset & \mathcal{N} \mathbin{\overline{\otimes}} R \\ \cup & & \cup \\ N_{\mathcal{F}} & \subset & \mathcal{N}_{\mathcal{F}} \mathbin{\overline{\otimes}} R \end{array} .$$

Remark furthermore that $N(\mathcal{N}_{\mathcal{F}} \mathbin{\overline{\otimes}} R)$ is dense in $\mathcal{N} \mathbin{\overline{\otimes}} R$. It follows that

$$W^*(\mathcal{N} \mathbin{\overline{\otimes}} R, e_N) = \overline{\operatorname{span}}^w \{x e_N y \mid x, y \in \mathcal{N}_{\mathcal{F}} \mathbin{\overline{\otimes}} R\}.$$

So we can identify the basic construction $\langle \mathcal{N}_{\mathcal{F}} \mathbin{\overline{\otimes}} R, e_{N_{\mathcal{F}}} \rangle$ for the inclusion $N_{\mathcal{F}} \subset \mathcal{N}_{\mathcal{F}} \mathbin{\overline{\otimes}} R$ with the von Neumann subalgebra of $\mathcal{N}_1$ generated by $\mathcal{N}_{\mathcal{F}} \mathbin{\overline{\otimes}} R$ and e_N. In particular, for $\mathcal{F} = \emptyset$, we get that the basic construction $P_1 := \langle P \mathbin{\overline{\otimes}} R, e_{\mathcal{L}Z} \rangle$ is isomorphic to the von Neumann subalgebra of $\mathcal{N}_1$ generated by $P \mathbin{\overline{\otimes}} R$ and e_N. Denote by $\| \cdot \|_2$ the 2-norm on $\mathrm{L}^2(\mathcal{N}_1)$ given by the semi-finite trace. Under this identification we have

$$\overline{\bigcup_{\mathcal{F} \subset \Gamma} \mathrm{L}^2(\langle \mathcal{N}_{\mathcal{F}} \mathbin{\overline{\otimes}} R, e_{N_{\mathcal{F}}} \rangle)}^{\| \cdot \|_2} = \mathrm{L}^2(\langle \mathcal{N} \mathbin{\overline{\otimes}} R, e_N \rangle) .$$

Note that $\langle P \mathbin{\overline{\otimes}} R, e_{\mathcal{L}Z} \rangle \subset \langle \mathcal{N}_{\mathcal{F}} \mathbin{\overline{\otimes}} R, e_{N_{\mathcal{F}}} \rangle$ for all $\mathcal{F} \subset \Gamma$. Denote by $E_{\mathcal{F}}$ the trace preserving conditional expectation of $\mathcal{N}_1$ onto $\langle \mathcal{N}_{\mathcal{F}} \mathbin{\overline{\otimes}} R, e_{N_{\mathcal{F}}} \rangle$ Choose $\varepsilon > 0$. Take a large enough finite subset $\mathcal{F} \subset \Gamma$ such that

$$\|T - E_{\mathcal{F}}(T)\|_2 < \varepsilon .$$

Since $T = \rho_g \sigma_g(T) \rho_g^*$, we get that $\|T - \rho_g \sigma_g(E_{\mathcal{F}}(T)) \rho_g^*\|_2 < \varepsilon$. As $\rho_g \in P \mathbin{\overline{\otimes}} R$, it follows that T lies at distance at most ε from $\langle \mathcal{N}_{\mathcal{F}g^{-1}} \mathbin{\overline{\otimes}} R, e_{N_{\mathcal{F}g^{-1}}} \rangle$. Since T also lies at distance at most ε from $\langle \mathcal{N}_{\mathcal{F}} \mathbin{\overline{\otimes}} R, e_{N_{\mathcal{F}}} \rangle$, we conclude that T lies at distance at most 2ε from $\langle \mathcal{N}_{\mathcal{F} \cap \mathcal{F}g^{-1}} \mathbin{\overline{\otimes}} R, e_{N_{\mathcal{F} \cap \mathcal{F}g^{-1}}} \rangle$ for all $g \in \Gamma$. We can choose g such that $\mathcal{F} \cap \mathcal{F}g^{-1} = \emptyset$ and conclude that T lies at distance at most 2ε from P_1. Since $\varepsilon > 0$ is arbitrary, it follows that $T \in P_1$.

So we can view T as the orthogonal projection of $\mathrm{L}^2(P \mathbin{\overline{\otimes}} R)$ onto a right $\mathcal{L}Z$ submodule of dimension $\tau(p)$. Since $pT = T$, the image of T is contained in $p\mathrm{L}^2(P \mathbin{\overline{\otimes}} R)$. Take projections $q_n \in \mathcal{L}Z$ with $\sum_n \tau(q_n) = \tau(p)$ and a right $\mathcal{L}Z$-linear isometry

$$\Theta : \bigoplus_{n \in \mathbb{N}} q_n \mathrm{L}^2(\mathcal{L}Z) \to p\mathrm{L}^2(P \mathbin{\overline{\otimes}} R)$$

onto the image of T. Denote by $w_n \in p\mathrm{L}^2(P \mathbin{\overline{\otimes}} R)$ the image under Θ of q_n sitting in position n. Note that

$$w_n = p w_n \quad , \quad E_{\mathcal{L}Z}(w_n^* w_m) = \delta_{n,m} q_n \quad \text{and} \quad T = \sum_{n \in \mathbb{N}} w_n e_{\mathcal{L}Z} w_n^*$$

where in the last formula we view T as an element of P_1. Identifying P_1 as above with the von Neumann subalgebra of $\mathcal{N}_1$ generated by $P \mathbin{\overline{\otimes}} R$ and e_N, it follows that

$$T = \sum_{n \in \mathbb{N}} w_n e_N w_n^* \, .$$

Since $T = w e_N w^*$, we get that $p e_N = \sum_n w^* w_n e_N w_n^* w$. Denote $x := e_N w_n^* w$. It follows that

$$|x|^2 = w^* w_n e_N w_n^* w = w^* w_n e_N w_n^* w e_N = |x|^2 e_N .$$

Then also $|x| = |x| e_N$ and hence $e_N w_n^* w = e_N w_n^* w e_N$. So $w^* w_n \in \mathrm{L}^2(\mathcal{N})$ preserves $\mathrm{L}^2(N)$. We conclude that $w^* w_n \in \mathrm{L}^2(N)$ for all n. It follows that $w_n^* w_m \in \mathrm{L}^1(N)$ for all n, m. Since also $w_n^* w_m \in \mathrm{L}^1(P \mathbin{\overline{\otimes}} R)$, we have $w_n^* w_m \in \mathrm{L}^1(\mathcal{L}Z)$. But then,

$$\delta_{n,m} q_n = E_{\mathcal{L}Z}(w_n^* w_m) = w_n^* w_m \, .$$

So, the elements w_n are partial isometries in $P \mathbin{\overline{\otimes}} R$ with mutually orthogonal left supports lying under p and with right supports equal to q_n. Since $\sum_n \tau(q_n) = \tau(p)$, we conclude that the formula

$$W := \sum_n e_{1,n} \otimes w_n$$

defines an element $W \in \mathrm{B}(\ell^2(\mathbb{N}), \mathbb{C}) \mathbin{\overline{\otimes}} P \mathbin{\overline{\otimes}} R$ satisfying $WW^* = p$, $W^* W = q$ where $q \in (\mathcal{L}Z)^\infty$ is the projection given by $q := \sum_n e_{nn} \otimes q_n$. We also have that $v := W^* w$ belongs to $\mathrm{B}(\mathbb{C}, \ell^2(\mathbb{N})) \mathbin{\overline{\otimes}} N$ and satisfies $v^* v = p$, $v v^* = q$.

Recall that $T = \rho_g \sigma_g(T) \rho_g^*$. Let $\gamma : \Gamma \to \mathcal{U}(q(\mathcal{L}Z)^\infty q)$ be the unique group homomorphism satisfying

$$\Theta(\gamma_g \xi) = \rho_g U_g \Theta(\xi) \quad \text{for all } \; g \in \Gamma, \xi \in \bigoplus_n q_n \mathrm{L}^2(\mathcal{L}Z)$$

where U_g is the unitary on $\mathrm{L}^2(P \mathbin{\overline{\otimes}} R)$ implementing the action σ_g on P_1. By construction $W \gamma_g = \rho_g \sigma_g(W)$ for all $g \in \Gamma$. Since $\omega_g = w^* \, \rho_g \, \sigma_g(w)$, we conclude that $\omega_g = v^* \, \gamma_g \, \sigma_g(v)$.

We finally prove that γ is unique up to unitary conjugacy. Assume that we also have a projection $q_1 \in (\mathcal{L}Z)^\infty$, a generalized 1-cocycle φ_g for the action $\Gamma \curvearrowright (\mathcal{L}Z)^\infty$ with support q_1 and a partial isometry $v_1 \in \mathrm{B}(\mathbb{C}, \ell^2(\mathbb{N})) \mathbin{\overline{\otimes}} N$ with $v_1^* v_1 = p$ and $v_1 v_1^* = q_1$, such that $\omega_g = v_1^* \, \varphi_g \, \sigma_g(v_1)$. Then the element $v_1 v^* \in q_1 N^\infty q$ satisfies

$$v_1 v^* = \varphi_g \sigma_g(v_1 v^*) \gamma_g^* \text{ for all } g \in \Gamma.$$

Denote by $\tilde{E}_{\mathcal{F}}$ the unique trace preserving conditional expectation $\tilde{E}_{\mathcal{F}} : N \to N_{\mathcal{F}}$. Choose $\epsilon > 0$ and denote by $\|\cdot\|_2$ the 2-norm on N^∞ given by the semi-finite trace $\mathrm{Tr}\otimes\tau$. Take a large enough finite subset $\mathcal{F} \subset \Gamma$ such that

$$\|v_1 v^* - (\mathrm{id}_{\mathrm{B}(\ell^2(\mathbb{N}))} \otimes \tilde{E}_{\mathcal{F}})(v_1 v^*)\|_2 < \epsilon \, .$$

Since $v_1 v^* = \varphi_g \sigma_g(v_1 v^*)\gamma_g^*$, we also get that

$$\|v_1 v^* - \varphi_g(\mathrm{id}_{\mathrm{B}(\ell^2(\mathbb{N}))} \otimes \sigma_g \circ \tilde{E}_{\mathcal{F}})(v_1 v^*)\gamma_g^*\|_2 < \epsilon \, .$$

Recall that $\varphi_g, \gamma_g^* \in (\mathcal{L}Z)^\infty$. Similarly as above we conclude that $v_1 v^*$ lies at distance at most 2ϵ from $(\mathcal{L}Z)^\infty$. Since $\epsilon > 0$ is arbitrary, it follows that $v_1 v^* \in (\mathcal{L}Z)^\infty$, providing the required unitary conjugacy between γ and φ. $\square$

3.3 Proof of Theorem 3.1

Remark 3.10. Let $\Gamma \overset{\alpha}{\curvearrowright} (X,\mu)$ and $\Lambda \overset{\beta}{\curvearrowright} (Y,\nu)$ be essentially free, ergodic, probability measure preserving actions of countable groups on standard measure spaces. Recall (see also 1.25) that a stable orbit equivalence between α and β is a measure space isomorphism $\Delta : X_0 \to Y_0$ between non-negligible subsets $X_0 \subset X, Y_0 \subset Y$, satisfying

$$\Delta(\Gamma \cdot x \cap X_0) = \Lambda \cdot \Delta(x) \cap Y_0$$

for a.e. $x \in X_0$. By ergodicity of $\Gamma \curvearrowright X$, we can choose a measurable map $\Theta : X \to X_0$ satisfying $\Theta(x) \in \Gamma \cdot x$ for a.e. $x \in X$. Denote $\Delta_0 := \Delta \circ \Theta$. By construction Δ_0 is a local isomorphism from X to Y. This means that $\Delta_0 : X \to Y$ is a Borel map and that X can be partitioned into a sequence of non-negligible subsets $\mathcal{W} \subset X$, such that the restriction of Δ_0 to any of these subsets $\mathcal{W}$ is a measure space isomorphism of $\mathcal{W}$ onto some non-negligible subset of Y. Also by construction Δ_0 is orbit preserving, meaning that for a.e. $x, y \in X$ we have that $x \in \Gamma \cdot y$ if and only if $\Delta_0(x) \in \Lambda \cdot \Delta_0(y)$.

Remark that for $\Gamma = \mathrm{SL}_n(\mathbb{Z}) *_\Sigma (\Sigma \times \Lambda)$ as in Notation 2.5, $\mathrm{SL}_n(\mathbb{Z})$ is an infinite rigid subgroup that is wq-normal in Γ. We now prove Theorem 3.1.

Proof of Theorem 3.1. Let p be a projection in M and let $\alpha : M \to pMp$ be a stable automorphism of M. We will prove that $p = 1$, i.e. α is an automorphism of M. This means that $\mathcal{F}(M) = \{1\}$.

Denote $K := \mathbb{Z}_p^n$ and $Z := \mathbb{Z}^n$. Recall that $X = K^\Gamma$ with Γ as in Notation 2.5 and that $M = \mathrm{L}^\infty(X) \rtimes (Z \rtimes \Gamma)$ where Z acts by translation after embedding by i and Γ acts by Bernoulli shift. Denote $B := \mathrm{L}^\infty(X) \rtimes Z$. Remark that $\mathcal{Z}(B) = \mathrm{L}^\infty(X/i(K))$. By Lemma 3.4 there exist projections $q, r \in B$ such that after composition of α with an inner automorphism of M, we have $\alpha(q) = r$ and $\alpha(qBq) = rBr$. Furthermore by Lemma 3.6, $\Psi_0 : \mathcal{Z}(B)z_B(q) \to \mathcal{Z}(B)z_B(r)$ given by the composition of

$$\mathcal{Z}(B)z_B(q) \quad \overset{\varphi_q}{\to} \quad \mathcal{Z}(qBq) \quad \overset{\alpha}{\to} \quad \mathcal{Z}(rBr) \quad \overset{\varphi_r^{-1}}{\to} \quad \mathcal{Z}(B)z_B(r)$$

is a stable orbit equivalence of the action $\Gamma \curvearrowright X/i(K)$. To simplify notation in the rest of the proof, we will use $z(q), z(r)$ for $z_B(q), z_B(r)$ respectively.

Denote by $\mathcal{Q}, \mathcal{R} \subset X/i(K)$ the support of $z(q), z(r)$ respectively. Let $\Delta_0 : \mathcal{R} \to \mathcal{Q}$ be the measure space isomorphism such that $\Psi_0(b) = b \circ \Delta_0$ for all $b \in Z(B)z(q) = \mathrm{L}^\infty(\mathcal{Q})$. By ergodicity of the action $\Gamma \curvearrowright X/i(K)$, we can extend Δ_0 to a local isomorphism from $X/i(K)$ to $X/i(K)$ that is orbit preserving, as explained in Remark 3.10.

Let G_1 be the locally compact second countable group $K \rtimes \Gamma$, having K as a compact open normal subgroup. By Lemma 3.7 the action $G_1 \overset{\sigma}{\curvearrowright} X$ where K acts by translation after embedding by i and Γ acts by Bernoulli shift, is $\mathcal{U}_{fin}$-cocycle superrigid. Remark that the restricted action $\sigma_{|K}$ is proper by compactness of K and that $G_1/K \curvearrowright X/i(K)$ is the action $\Gamma \curvearrowright X/i(K)$.

Since the action $\Gamma \curvearrowright X/i(K)$ is mixing, it is not induced from a proper subgroup by Lemma 1.8. Applying [PV08b, Lemma 5.10] to the stable orbit equivalence Δ_0 between the action $\Gamma \curvearrowright X/i(K)$ and itself, we find an open normal subgroup $K_1 \lhd G_1$ such that the following holds.

(i) The restricted action $\sigma_{|K_1}$ is proper.

(ii) The actions $G_1/K_1 \curvearrowright X/K_1$ and $\Gamma \curvearrowright X/i(K)$ are conjugate through a non-singular isomorphism $\Delta : X/K_1 \to X/i(K)$ and a group isomorphism $\delta : G_1/K_1 \to \Gamma$.

(iii) $\Delta_0(i(K) \cdot x) \in \Gamma \cdot \Delta(K_1 \cdot x)$ for almost all $x \in X$.

Since the restricted action $K_1 \curvearrowright X$ is essentially free and proper, there exists a measurable map $\pi : X \to K_1$ such that $\pi(k \cdot x) = k\pi(x)$ for almost all $(k, x) \in K_1 \times X$. Then the pushforward of the invariant probability measure on X is an invariant probability measure on K_1. So K_1 is compact. Set $K_2 := K \cap K_1$, then K_2 is an open normal subgroup of K and hence has finite index. By (ii) the group $\frac{G_1}{K_1}$ is isomorphic to Γ, so either $K_2 = K$ or $[K : K_2] = 2$. But the latter is impossible as K has no Γ-invariant normal subgroups of index 2. So $K_2 = K$. Since K_1 is a compact normal subgroup of $G_1 = K \rtimes \Gamma$, the image Γ_1 of K_1 in Γ is a finite normal subgroup of Γ. However the only finite normal subgroups in Γ are $\{1\}$ and $\{1,-1\}$. Assume $\Gamma_1 = \{-1,1\}$. Then $\frac{G_1}{K_1}$ is simple, but Γ is not, contradicting (ii). So we found that $K_1 = K$.

It follows that δ is a group automorphism of Γ and that $\Delta : X/i(K) \to X/i(K)$ is a δ-conjugacy between the action $\Gamma \curvearrowright X/i(K)$ and itself, satisfying

$$\Delta_0(i(K)\cdot x) \in \Gamma \cdot \Delta(i(K)\cdot x) \tag{3.3}$$

for almost all $x \in X$. Remark that Δ is a measure space isomorphism, as any conjugacy between ergodic probability measure preserving actions is measure preserving. In particular, the compression constant of Δ_0 is 1.

Then, by ergodicity of the action $\Gamma \curvearrowright X/i(K)$, one can build a measure space isomorphism $\tilde{\Delta}_0 : X/i(K) \to X/i(K)$ such that $\tilde{\Delta}_{0|\mathcal{R}} = \Delta_{0|\mathcal{R}}$ and $\tilde{\Delta}_0(i(K)\cdot x) \in \Gamma \cdot \Delta_0(i(K)\cdot x)$ for almost all $x \in X$. In particular, $\tilde{\Delta}_0$ still satisfies (3.3) for almost all $x \in X$.

It follows that there exists a unitary $u \in \mathrm{L}^\infty(X/i(K)) \rtimes \Gamma$ such that $u(b\circ\tilde{\Delta}_0)u^* = b\circ\Delta$ for all $b \in \mathrm{L}^\infty(X/i(K))$. Recall that the stable orbit equivalence $\varphi_r^{-1}\circ\alpha\circ\varphi_q$ of the action $\Gamma \curvearrowright X/i(K)$ is given by

$$(\varphi_r^{-1}\circ\alpha\circ\varphi_q)(b) = b\circ\Delta_0 = b\circ\tilde{\Delta}_0$$

for all $b \in Z(B)z(q)$. Replacing r by uru^* and α by $\operatorname{Ad} u\circ\alpha$, we then find that $(\varphi_r^{-1}\circ\alpha\circ\varphi_q)(b) = b\circ\Delta$ for all $b \in Z(B)z(q)$.

To summarize, we found a group automorphism δ of Γ and a δ-conjugacy Δ of the action $\Gamma \curvearrowright X/i(K)$ such that $(\varphi_r^{-1}\circ\alpha\circ\varphi_q)(b) = b\circ\Delta$ for all $b \in Z(B)z(q)$. For convenience, we denote $\Psi(b) = b\circ\Delta$ for all $b \in \mathrm{L}^\infty(X/i(K))$.

Let τ be the unique tracial state on M. We show that after composition with an inner automorphism of M, α satisfies

$$\alpha(b) = \Psi(b)p_0 \quad \text{for all} \quad b \in \mathcal{Z}(B)z(q)\ , \tag{3.4}$$

where p_0 is a projection in B of trace $\tau(p)\tau(z(q))$. The proof makes use of the following equality. For every projection $q_0 \leq q$ we have

$$E_{\mathcal{Z}(B)}(\alpha(q_0)) = \tau(p)\Psi(E_{\mathcal{Z}(B)}(q_0))\ . \tag{3.5}$$

Here $E_{\mathcal{Z}(B)}$ denotes the unique trace preserving conditional expectation of M onto $\mathcal{Z}(B)$. To see that (3.5) holds, first remark that $\alpha(xq) = \Psi(x)r$ for all $x \in \mathcal{Z}(B)$. Then use the fact that α is τ-scaling ($\tau\circ\alpha = \tau(p)\tau$) and that $E_{\mathcal{Z}(B)}$

and Ψ are τ-preserving to show that

$$\begin{aligned}\tau(\Psi(x)E_{\mathcal{Z}(B)}(\alpha(q_0))) &= \tau(\Psi(x)\alpha(q_0)) \\ &= \tau(\alpha(xq_0)) \\ &= \tau(p)\tau(xq_0) \\ &= \tau(p)\tau(E_{\mathcal{Z}(B)}(xq_0)) \\ &= \tau(p)\,\tau(\Psi(x)\Psi(E_{\mathcal{Z}(B)}(q_0)))\end{aligned}$$

for all $x \in \mathcal{Z}(B)$. Formula (3.5) follows.

We will show that there exist partial isometries $v_n, w_n \in B$ satisfying the following properties:

$$\begin{aligned} v_n^* v_n \leq q, \qquad\qquad & \alpha(v_n^* v_n) = w_n^* w_n, \\ \sum_n v_n v_n^* = z(q), \quad & w_n w_n^* \text{ mutually orthogonal.}\end{aligned} \tag{3.6}$$

Then define $p_0 := \sum w_n w_n^*$. Note that p_0 is a projection in B of trace $\tau(p)\tau(z(q))$. Define $v := \sum_n w_n \alpha(v_n^*) \in M$. Remark that $vv^* = p_0$ and $v^*v = \alpha(z(q))$. Extend v to a unitary $u \in M$. One verifies that $(\operatorname{Ad} u \circ \alpha)(b) = \Psi(b)p_0$ for all $b \in \mathcal{Z}(B)z(q)$ so that (3.4) is shown.

It remains to prove the existence of the partial isometries in (3.6). Consider the set

$$\begin{aligned}\mathcal{J} = \Big\{ \, \{ \, (v_n, w_n) \in B \times B \quad \Big| \quad & v_n^* v_n \leq q, \ \alpha(v_n^* v_n) = w_n^* w_n, \\ & v_n v_n^* \text{ mutually } \perp, \ w_n w_n^* \text{ mutually } \perp \, \} \, \Big\}.\end{aligned}$$

Remark that $\{(q, r)\} \in \mathcal{J}$. The set $\mathcal{J}$ is partially ordered by inclusion. By Zorn's lemma, take a maximal element $\{(v_n, w_n) \mid n\}$ of $\mathcal{J}$. It suffices to show that $\sum v_n v_n^* = z(q)$. Assume that $\sum v_n v_n^* < z(q)$. Then there exists a partial isometry $v \in B$ such that $v^*v \leq q$ and $vv^* \leq z(q) - \sum v_n v_n^*$. We claim that

$$\alpha(v^*v) \prec z(r) - \sum w_n w_n^* \,, \tag{3.7}$$

where $\prec$ refers to the comparison of projections in B. So there exists $w \in B$ such that $w^*w = \alpha(v^*v)$ and $ww^* \leq z(r) - \sum w_n w_n^*$. This means that we can add (v,w) to the family $\{(v_n,w_n) \mid n\}$, contradicting its maximality. It remains to prove (3.7). Using (3.5) we find that

$$E_{\mathcal{Z}(B)}(\alpha(v^*v)) = \tau(p)\Psi(E_{\mathcal{Z}(B)}(v^*v)) = \Psi(E_{\mathcal{Z}(B)}(\tau(p)vv^*)) \; .$$

On the other hand

$$\begin{aligned} E_{\mathcal{Z}(B)}\big(z(r) - \sum w_n w_n^*\big) &= \Psi(z(q)) - \tau(p)\Psi\big(E_{\mathcal{Z}(B)}(\sum_n v_n v_n^*)\big) \\ &= \Psi\big(E_{\mathcal{Z}(B)}(z(q) - \tau(p)\sum_n v_n v_n^*)\big) \; . \end{aligned}$$

Since $\tau(p)vv^* \leq z(q) - \tau(p)\sum v_n v_n^*$, it is clear that $E_{\mathcal{Z}(B)}(\alpha(v^*v)) \leq E_{\mathcal{Z}(B)}(z(r) - \sum w_n w_n^*)$ and (3.7) follows.

We now have that $\alpha(b) = \Psi(b)p_0$ for all $b \in \mathcal{Z}(B)z(q)$. Note that $E_{\mathcal{Z}(B)}(p_0) = \tau(p)z(r)$. Using ergodicity of the action $\Gamma \curvearrowright X/i(K)$, one builds a unitary in $u \in M$ such that after composition with $\mathrm{Ad}\, u$, α satisfies

$$\alpha(b) = \Psi(b)\tilde{p} \quad \text{for all} \quad b \in \mathcal{Z}(B) \; ,$$

where $\tilde{p}$ is a projection in B.

Denote by $(\sigma_g)_{g\in\Gamma}$ the action of Γ on $\mathrm{L}^\infty(X) \rtimes Z$, implemented by $\mathrm{Ad}\, u_g$ and corresponding to the Bernoulli action on $\mathrm{L}^\infty(X)$ and given by α_g on $\mathcal{L}Z$. Since the relative commutant of $\mathrm{L}^\infty(X/i(K))$ inside M equals $\mathrm{L}^\infty(X) \rtimes Z$, it follows that

$$\alpha(u_{\delta(g)}) = \omega_g u_g \tilde{p} \quad \text{for all } g \in \Gamma, \text{ where} \quad \omega_g \in \tilde{p}(\mathrm{L}^\infty(X) \rtimes Z)\sigma_g(\tilde{p}) \; .$$

Note that

$$\omega_g \omega_g^* = \tilde{p} \;\; , \;\; \omega_g^* \omega_g = \sigma_g(\tilde{p}) \quad \text{and} \quad \omega_{gh} = \omega_g \, \sigma_g(\omega_h) \;\; \text{for all} \;\; g,h \in \Gamma \; .$$

As was remarked before the proof of the theorem, Γ admits an infinite rigid subgroup that is wq-normal in Γ. Hence by Theorem 3.9 there exists a projection $q \in \mathrm{B}(\ell^2(\mathbb{N})) \mathbin{\overline{\otimes}} \mathcal{L}Z$ with $(\mathrm{Tr} \otimes \tau)(q) = \tau(\tilde{p})$, a partial isometry $v \in \mathrm{B}(\mathbb{C}, \ell^2(\mathbb{N})) \mathbin{\overline{\otimes}} (\mathrm{L}^\infty(X) \rtimes Z)$ and a family of partial isometries $\gamma_g \in q(\mathrm{B}(\ell^2(\mathbb{N})) \mathbin{\overline{\otimes}} \mathcal{L}Z)\sigma_g(q)$ satisfying $\gamma_g\gamma_g^* = q, \gamma_g^*\gamma_g = \sigma_g(q)$ and $\gamma_{gh} = \gamma_g\sigma_g(\gamma_h)$ such that

$$v^*v = \tilde{p} \;\; , \;\; vv^* = q \quad \text{and} \quad \omega_g = v^* \, \gamma_g \, \sigma_g(v) \;\; \text{for all} \;\; g \in \Gamma \; .$$

Denote by E the map

$$E := \mathrm{Tr} \otimes \mathrm{id} : \mathrm{B}(\ell^2(\mathbb{N})) \mathbin{\overline{\otimes}} \mathcal{L}Z \to \mathcal{L}Z.$$

Since q and $\sigma_g(q)$ are equivalent in $\mathrm{B}(\ell^2(\mathbb{N}))\mathbin{\overline{\otimes}}\mathcal{L}Z$, we get that $E(q)=\sigma_g(E(q))$ for all $g\in\Gamma$. Remark that $(\mathcal{L}Z,\tau)$ is isomorphic to $(\mathrm{L}^\infty(\mathbb{T}^n),\lambda)$ where λ denotes the Lebesgue measure on $\mathbb{T}^n$. By ergodicity of the action $\Gamma \curvearrowright \mathbb{T}^n$, it follows that $E(q)$ is equal to a constant $C\neq 0$. Viewing q as a measurable function on $\mathbb{T}^n$ that takes values in the projections of $\mathrm{B}(\ell^2(\mathbb{N}))$, we have that $\mathrm{Tr}(q(x))=C$ for almost all x in $\mathbb{T}^n$. In particular, $\mathrm{Tr}(q(x))\geq 1$ almost everywhere. Integrating over $\mathbb{T}^n$, we find that $(\mathrm{Tr}\otimes\tau)(q)\geq 1$. Because $(\mathrm{Tr}\otimes\tau)(q)=\tau(\tilde{p})\leq 1$, we must have $\tilde{p}=1$. This means that α is an automorphism of M. $\square$

Chapter 4

. . . containing an equivalence relation with non-trivial fundamental group

In this chapter we show that M contains a second Cartan subalgebra and calculate the fundamental group of the associated equivalence relation. In section §1 we show that the example satisfies the setting of Lemma 2.1, and describe the associated equivalence relation. In section §2 we use a result from [PV08b] to calculate its fundamental group.

4.1 M has a Cartan subalgebra that is non-conjugate to $\mathrm{L}^\infty(X)$

Theorem 4.1. *With $G \curvearrowright X$ as in Notation 2.5, $\mathrm{L}^\infty\left(\frac{(\mathbb{Z}_p^n)^\Gamma}{i(\mathbb{Z}_p^n)}\right) \overline{\otimes} \mathcal{L}\mathbb{Z}^n$ is a Cartan subalgebra of $M = \mathrm{L}^\infty(X) \rtimes G$ and the associated equivalence relation is equal to the orbit equivalence relation of the action $\widehat{\mathbb{Z}_p^n} \rtimes \Gamma \curvearrowright \mathbb{T}^n \times \frac{(\mathbb{Z}_p^n)^\Gamma}{i(\mathbb{Z}_p^n)}$ where $\widehat{\mathbb{Z}_p^n}$ acts on $\mathbb{T}^n = \widehat{\mathbb{Z}^n}$ by translation and Γ acts on both factors in the natural way.*

We will prove this theorem using the more general Lemma 2.1.

Proof of Theorem 4.1. Let $\Gamma = \mathrm{SL}_n(\mathbb{Z}) \underset{\Sigma}{*} (\Sigma \times \Lambda)$. We apply Lemma 2.1 where $\Gamma \curvearrowright \mathcal{Z} = (\mathbb{Z}_p^n)^\Gamma$ and $Z = Z_0 = \mathbb{Z}^n$, where we embed $\mathbb{Z}^n$ into $(\mathbb{Z}_p^n)^\Gamma$ by $i : \mathbb{Z}^n \to (\mathbb{Z}_p^n)^\Gamma : z \mapsto (\alpha_g(z))_{g\in\Gamma}$.

As $\overline{\mathbb{Z}^n} = \mathbb{Z}_p^n$ we see that indeed $\overline{\mathbb{Z}^n} \cap \mathbb{Z}^n = \mathbb{Z}^n$. Furthermore for any $g \in \Gamma$ the set $\{z - \alpha_g(z) \mid z \in \mathbb{Z}^n\}$ is either infinite or trivial. Indeed, if $x \in \{z - \alpha_g(z) \mid z \in \mathbb{Z}^n\}$ then all integer multiples of x are also in this set. Set

$$\Gamma_0 := \{g \in \Gamma \mid \alpha_g(z) = z \text{ for all } z \in \mathbb{Z}^n\}.$$

Let $g \in \Gamma_0\backslash\{e\}$. It is clear that $\{x \in \frac{(\mathbb{Z}_p^n)^\Gamma}{i(\mathbb{Z}_p^n)} \mid \alpha_g(x) = x\}$ has infinite index in $\frac{(\mathbb{Z}_p^n)^\Gamma}{i(\mathbb{Z}_p^n)}$.

So the conditions of Lemma 2.1 are satisfied and $\mathrm{L}^\infty\left(\frac{(\mathbb{Z}_p^n)^\Gamma}{i(\mathbb{Z}_p^n)}\right) \overline{\otimes}\, \mathcal{L}\mathbb{Z}^n = \mathrm{L}^\infty(\frac{(\mathbb{Z}_p^n)^\Gamma}{i(\mathbb{Z}_p^n)} \times \mathbb{T}^n)$ is a Cartan subalgebra of M. The associated equivalence relation is given by the action $\widehat{\mathbb{Z}_p^n} \rtimes \Gamma \curvearrowright \mathbb{T}^n \times \frac{(\mathbb{Z}_p^n)^\Gamma}{i(\mathbb{Z}_p^n)}$ where $\widehat{\mathbb{Z}_p^n}$ only acts on $\mathbb{T}^n = \widehat{\mathbb{Z}^n}$ and Γ acts both on $\mathbb{T}^n$ and $\frac{(\mathbb{Z}_p^n)^\Gamma}{i(\mathbb{Z}_p^n)}$. □

From now on we denote by $\mathcal{R}_2$ the equivalence relation associated with the Cartan subalgebra $\mathrm{L}^\infty(\frac{(\mathbb{Z}_p^n)^\Gamma}{i(\mathbb{Z}_p^n)}) \overline{\otimes} \mathcal{L}(\mathbb{Z}^n)$ of M.

4.2 The fundamental group of $\mathcal{R}_2$ is non-trivial

We prove that the fundamental group of the equivalence relation given by Theorem 4.1 is non-trivial and can be explicitly computed. It follows from Theorem 4.1 that $\mathcal{R}_2$ is the orbit equivalence relation of the action $\widehat{\mathbb{Z}_p^n} \rtimes \Gamma \curvearrowright \mathbb{T}^n \times \frac{(\mathbb{Z}_p^n)^\Gamma}{i(\mathbb{Z}_p^n)}$, where $\widehat{\mathbb{Z}_p^n}$ acts only on $\mathbb{T}^n = \widehat{\mathbb{Z}^n}$ by translation and Γ acts on $\mathbb{T}^n$ through the quotient map q and on $\frac{(\mathbb{Z}_p^n)^\Gamma}{i(\mathbb{Z}_p^n)}$ by Bernoulli shift. We will compute its fundamental group using [PV08b, Lemma 5.10]. Therefore we write the action as a quotient action $\frac{\tilde{G}}{N} \curvearrowright \frac{\tilde{X}}{N}$.

Recall that $q : \Gamma \to \mathrm{SL}_n(\mathbb{Z})$ denotes the quotient map. Let Γ act on $\mathbb{Z}[\frac{1}{p}]^n$ and on $\mathbb{Z}_p^n$ through q. Denote

$$\tilde{G} = \left(\mathbb{Z}[\frac{1}{p}]^n \times \mathbb{Z}_p^n\right) \rtimes \Gamma \quad \text{and} \quad \tilde{X} = \mathbb{R}^n \times (\mathbb{Z}_p^n)^\Gamma .$$

We defined $i : \mathbb{Z}_p^n \to (\mathbb{Z}_p^n)^\Gamma : z \mapsto (g \cdot z)_{g\in\Gamma}$. Let $\tilde{G} \curvearrowright \tilde{X}$ where $\mathbb{Z}[\frac{1}{p}]^n \times \mathbb{Z}_p^n$ acts by translation after embedding $\mathbb{Z}_p^n$ by i and Γ acts on $\mathbb{R}^n$ through q and on $(\mathbb{Z}_p^n)^\Gamma$ by Bernoulli shift. Before we prove the lemma we recall a few definitions.

Definition 4.2. The non-singular action $G \curvearrowright (X,\mu)$ of the l.c.s.c. group G on the standard measure space (X,μ) is called essentially free and proper if there exists a measurable map $\pi : X \to G$ such that $\pi(g \cdot x) = g\pi(x)$ for almost all $(g,x) \in G \times X$.

Remark that this actually says we can view $G \curvearrowright X$ as $G \curvearrowright X/G \times G$ where G only acts on G and X/G denotes the space of ergodic components of $G \curvearrowright X$. Indeed, it suffices to consider the map $\theta : X \to G \times X/G : x \mapsto (\pi(x), \overline{x})$. Furthermore one checks that θ conjugates the action $G \curvearrowright X$ with $\rho_g \times \mathrm{id}$ on $G \times X/G$, where ρ_g denotes right translation by g.

Note that if $G \curvearrowright X$ is essentially free and proper and $N \lhd G$, then $N \curvearrowright X$ is also essentially free and proper. This is immediately clear, as $G \cong \frac{G}{N} \times N$, by using a selector.

Now assume $G \curvearrowright (X,\mu)$ is an essentially free and proper action, and G a unimodular locally compact second countable group. We have an isomorphism $q \times \pi : X \to X/G \times G$, where $\pi : X \to G$ satisfies $\pi(g \cdot x) = g\pi(x)$ for almost all $x \in X$ and all $g \in G$, and q is the quotient map. Then $(q_*)(\mu)$ is a G-invariant measure on X/G and $(\pi_*)(\mu)$ is a G-invariant measure on G (and hence the Haar measure). In particular, if G is countable, the action has a fundamental domain. Remark that this implies that a Bernoulli action $G \curvearrowright X^G$, with X a standard probability space, can only be essentially free and proper if G is compact. Otherwise X^Γ would not have finite measure. This gives us a way to define finite covolume in this context.

Definition 4.3. Let $G \curvearrowright X$ be an essentially free and proper action. We say that $G \curvearrowright X$ has finite covolume if the measure on X/G is finite.

Finally we recall what is meant by an s-malleable action.

Definition 4.4. [Po03, §1.5] Let G be a locally compact second countable group and $G \curvearrowright (X,\mu)$ a Borel action preserving the finite or infinite measure μ. The action is called s-malleable if there exists

- a one-parameter group $(\alpha_t)_{t\in\mathbb{R}}$ of measure preserving transformations of $X \times X$,
- an involutive measure preserving transformation β of $X \times X$,

such that

- α_t and β commute with the diagonal action $G \curvearrowright X \times X$,
- $\alpha_1(x,y) \in \{y\} \times X$ for almost all $(x,y) \in X \times X$,
- $\beta(x,y) \in \{x\} \times X$ for almost all $(x,y) \in X \times X$,
- $\alpha_t \circ \beta = \beta \circ \alpha_{-t}$ for all $t \in \mathbb{R}$.

Example 4.5.

- A generalized Bernoulli action $\Gamma \curvearrowright (X_0^I, \mu)$ of a countable discrete group Γ acting on a countable set I, and with diffuse base space X_0 is s-malleable (see [Po03, §1.6] or [Po05, Lemma 4.5])
- The action $\mathrm{SL}_n(\mathbb{Z}) \curvearrowright \mathrm{M}_{n,k}(\mathbb{R})$ by left multiplication is s-malleable (see also [PV08b, Proof of theorem 1.3]). Indeed: for all $t \in \mathbb{R}$ and $X, Y \in \mathrm{M}_{n,k}(\mathbb{R})$ set

$$\begin{aligned}\alpha_t(X,Y) &= \left(\cos(\frac{\pi t}{2})A + \sin(\frac{\pi t}{2})B, -\sin(\frac{\pi t}{2})A + \cos(\frac{\pi t}{2})B\right)\\ \beta(X,Y) &= (X, -Y)\end{aligned}$$

  Easy calculations show that this indeed satisfies all conditions.

Note that whenever $G \curvearrowright X$ and $G \curvearrowright Y$ are s-malleable, so is the diagonal action $G \curvearrowright X \times Y$. This is immediately clear by taking direct products.

Set $N = \mathbb{Z}^n \times \mathbb{Z}_p^n$. Then $N \lhd \tilde{G}$ is an open normal subgroup, as we consider $\mathbb{Z}[\frac{1}{p}]^n$ with the discrete topology. The restricted action $N \curvearrowright \tilde{X}$ is essentially free and proper: consider

$$\pi : \mathbb{R}^n \times (\mathbb{Z}_p^n)^\Gamma \to \mathbb{Z} \times \mathbb{Z}_p^n : (x, (a_g)_g) \mapsto (\lfloor x \rfloor, a_e).$$

Observe that $\left(\frac{\tilde{G}}{N} \curvearrowright \frac{\tilde{X}}{N}\right) = \left(\widehat{\mathbb{Z}}_p^n \rtimes \Gamma \curvearrowright \mathbb{T}^n \times \frac{(\mathbb{Z}_p^n)^\Gamma}{i(\mathbb{Z}_p^n)}\right)$. This follows from the fact that $\frac{\mathbb{Z}[\frac{1}{p}]}{\mathbb{Z}} = \widehat{\mathbb{Z}_p}$ is indeed the Prüfer p-group.

Lemma 4.6. *$\tilde{G} \curvearrowright \tilde{X}$ is $\mathcal{U}_{fin}$-cocycle superrigid.*

Proof. We first prove that $\mathrm{SL}_n(\mathbb{Z}) \curvearrowright \tilde{X}$ is $\mathcal{U}_{fin}$-cocycle superrigid, using [PV08b, Theorem 5.3]. So we need to prove that $\mathrm{SL}_n(\mathbb{Z}) \curvearrowright \tilde{X}$ is s-malleable, that the diagonal action $\mathrm{SL}_n(\mathbb{Z}) \curvearrowright \tilde{X}^2$ has property (T) and that the 4-fold diagonal action $\mathrm{SL}_n(\mathbb{Z}) \curvearrowright \tilde{X}^4$ is ergodic.

As $\mathrm{SL}_n(\mathbb{Z}) \curvearrowright (\mathbb{Z}_p^n)^\Gamma$ is a Bernoulli action with diffuse base space, it follows from Example 4.5 that $\mathrm{SL}_n(\mathbb{Z}) \curvearrowright \tilde{X}$ is s-malleable.

Next we prove that $\mathrm{SL}_n(\mathbb{Z}) \curvearrowright \tilde{X}^2$ has property (T). This is very similar to the proof of [PV08b, Lemma 5.6]. Denote by $(e_i)_{i=1,\dots,n}$ the standard basis vectors in $\mathbb{R}^n$. The orbit of $(e_1, e_2) \in \mathbb{R}^n \times \mathbb{R}^n$ under the diagonal $\mathrm{SL}_n(\mathbb{R})$-action has complement of measure zero. Hence we can identify $\mathrm{SL}_n(\mathbb{Z}) \curvearrowright \tilde{X}^2$ with

$$\mathrm{SL}_n(\mathbb{Z}) \curvearrowright \mathrm{SL}_n(\mathbb{R})/H_0 \times X \times X$$

where $H_0 = \mathrm{Stab}_{\mathrm{SL}_n(\mathbb{R})}(e_1, e_2)$. Note that

$$H_0 \cong \mathrm{SL}_{n-2}(\mathbb{R}) \ltimes \mathrm{M}_{n-2,2}(\mathbb{R}).$$

Now by [PV08b, Proposition 3.5] $\mathrm{SL}_n(\mathbb{Z}) \curvearrowright \tilde{X}^2$ has property (T) if and only if $H_0 \curvearrowright \mathrm{SL}_n(\mathbb{Z})\backslash\mathrm{SL}_n(\mathbb{R}) \times X \times X$ has property (T). But $\mathrm{SL}_n(\mathbb{Z})\backslash\mathrm{SL}_n(\mathbb{R}) \times X \times X$ is a probability space. Hence if H_0 has property (T), so will the action

$$H_0 \curvearrowright \mathrm{SL}_n(\mathbb{Z})\backslash\mathrm{SL}_n(\mathbb{R}) \times X \times X$$

by [PV08b, Proposition 3.2]. Now H_0 has property (T) for $n - 2 \geq 3$ (see example 1.53).

We still need to show that $\mathrm{SL}_n(\mathbb{Z}) \curvearrowright \tilde{X}^4$ is ergodic. From [PV08b, Lemma 5.6] we know that $\mathrm{SL}_n(\mathbb{Z}) \curvearrowright (\mathbb{R}^n)^4$ is ergodic. In fact this action is properly ergodic, as $\mathrm{SL}_n(\mathbb{Z})$-orbits are countable and hence negligible. Now $\mathrm{SL}_n(\mathbb{Z}) \curvearrowright (\mathbb{Z}_p^n)^\Gamma$ is strongly mixing and hence mildly mixing. It follows from Proposition 1.13 that $\mathrm{SL}_n(\mathbb{Z}) \curvearrowright \tilde{X}^4$ is ergodic.

So it follows from [PV08b, Theorem 5.3] that $\mathrm{SL}_n(\mathbb{Z}) \curvearrowright \mathbb{R}^n \times (\mathbb{Z}_p^n)^\Gamma$ is $\mathcal{U}_{fin}$-cocycle superrigid.

Now let

$$\omega : \left((\mathbb{Z}[\frac{1}{p}] \times \mathbb{Z}_p)^n \rtimes \Gamma \right) \times (\mathbb{R}^n \times (\mathbb{Z}_p^n)^\Gamma) \to \mathcal{G}$$

be a 1-cocycle for the action $\tilde{G} \curvearrowright \tilde{X}$ with values in a Polish group of finite type. By the previous paragraphs, we may assume that $\omega_{|\mathrm{SL}_n(\mathbb{Z})}$ is a group morphism. Let $a \in \mathbb{Z}[\frac{1}{p}]$, $b \in \mathbb{Z}_p$ be any two elements. Write

$$(a, b)_i = \left(((0,0), \dots, (0,0), (a,b), (0,0), \dots, (0,0))^T, e \right) \in (\mathbb{Z}[\frac{1}{p}] \times \mathbb{Z}_p)^n \rtimes \Gamma$$

where we write (a, b) in the i^{th} row. Then $\mathrm{SL}_n(\mathbb{Z}) \cap (a,b)_i \mathrm{SL}_n(\mathbb{Z}) (a,b)_i^{-1} = H_i$, where H_i is the group of matrices in $\mathrm{SL}_n(\mathbb{Z})$ that leave all vectors $(a, b)_i$ invariant. One easily sees that this means the i^{th} column of the matrix has to be the i^{th} unit vector. Note that H_i is isomorphic to $\mathrm{SL}_{n-1}(\mathbb{Z}) \ltimes \mathbb{Z}^{n-1}$. Indeed, note that

$H_i \cong H_j$ for all i, j and consider

$$\psi : H_1 \to \mathrm{SL}_{n-1}(\mathbb{Z}) \ltimes \mathbb{Z}^{n-1} : \begin{pmatrix} 1 & a_1 & \dots & a_{n-1} \\ 0 & & & \\ \vdots & & B & \\ 0 & & & \end{pmatrix} \mapsto (B, (a_1, \dots, a_{n-1})).$$

One checks that this is indeed a group isomorphism.

By Lemma 1.60 it now suffices to prove that $H_i \curvearrowright \tilde{X}^2$ is ergodic for all i to get that ω is a group morphism on $(\mathbb{Z}[\frac{1}{p}]^n \times \mathbb{Z}_p^n) \rtimes \mathrm{SL}_n(\mathbb{Z})$. But as H_i acts mixingly on $(\mathbb{Z}_p^n)^\Gamma$, by Proposition 1.13 it suffices to prove that $H_i \curvearrowright (\mathbb{R}^n)^2$ is properly ergodic.

We show that $H_1 \curvearrowright (\mathbb{R}^n)^2$ is properly ergodic. Let $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be an H_1-invariant function. For all $x_1, x_{n+1} \in \mathbb{R}$, the map

$$F_{x_1,x_{n+1}} : \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \to \mathbb{R} : (x_2, \dots, x_n, x_{n+2}, \dots, x_{2n}) \mapsto F(x_1, \dots, x_{2n})$$

is $\mathrm{SL}_{n-1}(\mathbb{Z})$-invariant. By ergodicity of the diagonal action $\mathrm{SL}_{n-1}(\mathbb{Z}) \curvearrowright (\mathbb{R}^{n-1})^2$ (see [PV08b, Lemma 5.6]) we find that $F_{x_1,x_{n+1}}$ is essentially constant for all $x_1, x_{n+1} \in \mathbb{R}$, say $c_{x_1,x_{n+1}}$. Set

$$E : \mathbb{R}^2 \to \mathbb{R} : (x_1, x_{n+1}) \mapsto c_{x_1,x_{n+1}}.$$

Then for almost all $(x_1, x_{n+1}) \in \mathbb{R}^2$, for almost all $(x_2, \dots, x_n, x_{n+2}, \dots, x_{2n}) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$ and for all $a_2, \dots, a_n \in \mathbb{Z}$ we have

$$\begin{aligned} & E(x_1, x_{n+1}) \\ = \;& F(x_1, \dots, x_{2n}) \\ = \;& F(x_1 + a_2 x_2 + \dots + a_n x_n, x_2, \dots, x_n, \\ & x_{n+1} + a_2 x_{n+2} + \dots + a_n x_{2n}, x_{n+2}, \dots, x_{2n}) \\ = \;& E(x_1 + a_2 x_2 + \dots + a_n x_n, x_{n+1} + a_2 x_{n+2} + \dots + a_n x_{2n}) \,. \end{aligned}$$

Hence E is essentially constant. It follows that F is essentially constant. Furthermore it is clear that any H_1-orbit is negligible (they are countable). So $H_1 \curvearrowright (\mathbb{R}^n)^2$ is properly ergodic. The same reasoning holds for all i, so that ω is a group morphism on $(\mathbb{Z}[\frac{1}{p}]^n \times \mathbb{Z}_p^n) \rtimes \mathrm{SL}_n(\mathbb{Z})$.

In particular ω is a group morphism on $(\mathbb{Z}[\frac{1}{p}]^n \times \mathbb{Z}_p^n) \rtimes \Sigma$ and this commutes with Λ. To conclude that $(\mathbb{Z}[\frac{1}{p}]^n \times \mathbb{Z}_p^n) \rtimes \Gamma \curvearrowright \mathbb{R}^n \times (\mathbb{Z}_p^n)^\Gamma$ is $\mathcal{U}_{fin}$-cocycle superrigid, it suffices to prove that the diagonal action

$$(\mathbb{Z}[\frac{1}{p}]^n \times \mathbb{Z}_p^n) \rtimes \Sigma \curvearrowright \mathbb{R}^n \times \mathbb{R}^n \times ((\mathbb{Z}_p^n)^\Gamma)^2$$

is ergodic. Let $F : \mathbb{R}^n \times \mathbb{R}^n \times ((\mathbb{Z}_p^n)^\Gamma)^2 \to \mathbb{R}$ be a $(\mathbb{Z}[\frac{1}{p}]^n \times \mathbb{Z}_p^n) \rtimes \Sigma$-invariant function. As $\mathbb{Z}[\frac{1}{p}]^n$ is dense in $\mathbb{R}^n$, F satisfies $F(x, y, z) = H(x - y, z)$ where $H : \mathbb{R}^n \times ((\mathbb{Z}_p^n)^\Gamma)^2 \to \mathbb{R}$ is $\mathbb{Z}_p^n \rtimes \Sigma$-invariant.

In particular, H is Σ-invariant. As Σ acts on $((\mathbb{Z}_p^n)^\Gamma)^2$ mildly mixingly, it suffices to see that $\Sigma \curvearrowright \mathbb{R}^n$ is ergodic. We prove this by induction on m where $2m = n$. To this end, set Σ_m to be m copies of $\mathrm{SL}_2(\mathbb{Z})$ on the diagonal.

> $\mathbf{m = 1}$: If $m = 1$ we simply get $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{R}^2$ which is known to be ergodic.
>
> $\mathbf{m - 1 \to m}$: Assume $\Sigma_{m-1} \curvearrowright \mathbb{R}^{n-2}$ is ergodic. We prove that $\Sigma_m \curvearrowright \mathbb{R}^n$ is ergodic. Let $F : \mathbb{R}^n \to \mathbb{R}$ be a Σ_m-invariant function. Let $F_{(a,b)} : \mathbb{R}^{n-2} \to \mathbb{R} : (x_1, \ldots, x_{n-2}) \mapsto F(a, b, x_1, \ldots, x_{m-2})$ then for all (a, b) $F_{(a,b)}$ is a constant function, and hence we found an $\mathrm{SL}_2(\mathbb{Z})$-invariant function $G : \mathbb{R}^2 \to \mathbb{R}$, which has to be constant.

So we have shown that $(\mathbb{Z}[\frac{1}{p}]^n \times \mathbb{Z}_p^n) \rtimes \Gamma \curvearrowright \mathbb{R}^n \times (\mathbb{Z}_p^n)^\Gamma$ is $\mathcal{U}_{fin}$-cocycle superrigid. □

Now that we have proven that $(\mathbb{Z}[\frac{1}{p}]^n \times \mathbb{Z}_p^n) \rtimes \Gamma \curvearrowright \mathbb{R}^n \times (\mathbb{Z}_p^n)^\Gamma$ is $\mathcal{U}_{fin}$-cocycle superrigid, we can use [PV08b, Lemma 5.10] to calculate the fundamental group. However, we will need the following two lemmas. The first gives us an easier way to describe Γ-invariant subgroups of $\mathbb{Z}^n$. The second lemma describes the Γ-invariant subgroups of $\mathbb{Z}[\frac{1}{p}]^n$.

Lemma 4.7. *Let $n \geq 2$. For all $(\mu_1, \ldots, \mu_n)^T \in \mathbb{Z}^n$, we find*

$$\{(z, 0, \ldots, 0)^T, (0, z, \ldots, 0)^T, \ldots, (0, 0, \ldots, z)^T\} \subset \mathrm{SL}_n(\mathbb{Z}) \cdot (\mu_1, \ldots, \mu_n)^T ,$$

where $z = \gcd(\mu_1, \ldots, \mu_n)$.

Proof. We prove this by induction on n.

- Let $n = 2$. Write $d = \gcd(\mu_1, \mu_2)$, then there are $a, b \in \mathbb{Z}$ such that $a\mu_1 + b\mu_2 = d$. Note that necessarily $\gcd(a, b) = 1$. By Bézouts identity we find c, e such that $ae - bc = 1$. But then

$$\begin{pmatrix} a & b \\ c & e \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} d \\ c\mu_1 + e\mu_2 \end{pmatrix} .$$

Now $c\mu_1 + e\mu_2 = kd$ for some $k \in \mathbb{Z}$. But then

$$\begin{pmatrix} 1 & 0 \\ k-1 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & e \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} d \\ 0 \end{pmatrix}.$$

Similarly we can find $g \in \mathrm{SL}\,2$ such that $g \cdot \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} 0 \\ d \end{pmatrix}$.

- Now assume the claim holds for $n-1$. Write $d_{n-1} = \gcd(\mu_1, \dots, \mu_{n-1})$, then $d = \gcd(\mu_1, \dots, \mu_n) = \gcd(d_{n-1}, \mu_n)$. Note that for all $g \in \mathrm{SL}_{n_1}(\mathbb{Z})$ we have $\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_n(\mathbb{Z})$, and $\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} e_n = e_n$. So we can use the induction hypothesis to find that

$$\left\{ \begin{pmatrix} d_{n-1} \\ 0 \\ \vdots \\ 0 \\ \mu_n \end{pmatrix}, \begin{pmatrix} 0 \\ d_{n-1} \\ \vdots \\ 0 \\ \mu_n \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ d_{n-1} \\ \mu_n \end{pmatrix} \right\} \subset \Gamma \cdot \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix}.$$

  Now there are $a, b \in \mathbb{Z}$ such that $ad_{n-1} + b\mu_1 = d$ where a, b are coprime. This is clearly exactly the same as the basic case, so the lemma is proven.

$\square$

We now describe the $\mathrm{SL}_n(\mathbb{Z})$-invariant subgroups of $\mathbb{Z}[\frac{1}{p}]^n$.

Lemma 4.8. *If $G < \mathbb{Z}[\frac{1}{p}]^n$ is an $\mathrm{SL}_n(\mathbb{Z})$-invariant subgroup then either $G = \{0\}, G = (a\mathbb{Z})^n$ for some $a \in \mathbb{Z}[\frac{1}{p}]$ or $G = (b\mathbb{Z}[\frac{1}{p}])^n$ for some $b \in \mathbb{Z}$.*

Proof. Suppose $G \neq \{0\}$. If G is finitely generated, set $G = \langle a_1, \dots, a_m \rangle$. Choose $l \in \mathbb{N}$ such that

$$a_i = (\frac{a_{i1}}{p^l}, \dots, \frac{a_{in}}{p^l})^T$$

with $a_{i,j} \in \mathbb{Z}$ for all i, j. By Lemma 4.7, $G = (\frac{a}{p^l}\mathbb{Z})^n$ with $a = \gcd_{i,j}(a_{ij})$.

Now suppose G is not finitely generated. Then for each $k \in \mathbb{N}$ we find $m_k \geq k$ and $(a_{k1}, \dots, a_{kn}) \in \mathbb{Z}^n$ such that $p \nmid \gcd\{a_{k1}, \dots, a_{kn}\}$ and $\left(\frac{a_{k1}}{p^{m_k}}, \dots, \frac{a_{kn}}{p^{m_k}}\right)^T \in G$. By Lemma 4.7 we get, for $a_k := \gcd\{a_{k1}, \dots, a_{kn}\}$ that $\left(\frac{a_k}{p^{m_k}}, \dots, \frac{a_k}{p^{m_k}}\right)^T \in G$. Furthermore we can assume that $|a| < |a_k|$ implies $\left(\frac{a}{p^{m_k}}, \dots, \frac{a}{p^{m_k}}\right)^T \notin G$ or $p \mid a$. Set $b_k = \gcd_{i \leq k}(a_i)$. Let $b = \lim_{k \to \infty} b_k$. One verifies that $G = (b\mathbb{Z}[\frac{1}{p}])^n$ and $b \in \mathbb{Z}$. $\square$

Theorem 4.9. *Let $G \curvearrowright X$ as in Notation 2.5. Denote by $\mathcal{R}_2$ the equivalence relation associated with the Cartan subalgebra $\mathrm{L}^\infty(\frac{(\mathbb{Z}_p^n)^\Gamma}{i(\mathbb{Z}_p^n)}) \overline{\otimes} \mathcal{L}(\mathbb{Z}^n)$ of M. Then $\mathcal{F}(\mathcal{R}_2) = \{p^{kn} \mid k \in \mathbb{Z}\}$.*

Proof. To prove this we use [PV08b, Lemma 5.10]. Recall that

$$N = \mathbb{Z}^n \times \mathbb{Z}_p^n, \quad \tilde{G} = \left(\mathbb{Z}[\frac{1}{p}]^n \times \mathbb{Z}_p^n\right) \rtimes \Gamma \quad \text{and} \quad \tilde{X} = \mathbb{R}^n \times (\mathbb{Z}_p^n)^\Gamma .$$

.

Let $\Delta : \frac{\tilde{X}}{N} \to \frac{\tilde{X}}{N}$ be a stable orbit equivalence between $\frac{\tilde{G}}{N} \overset{\alpha}{\curvearrowright} \frac{\tilde{X}}{N}$ and itself. By [PV08b, Lemma 5.10], there exists

- a subgroup $\Lambda_0 < \frac{\tilde{G}}{N}$ and a non-negligible subset $Y_0 \subset \frac{\tilde{X}}{N}$, such that α is induced from $\Lambda_0 \curvearrowright Y_0$,
- an open normal subgroup $N_1 \lhd \tilde{G}$ such that the restricted action $N_1 \curvearrowright \tilde{X}$ is proper,

such that $\frac{\tilde{G}}{N_1} \curvearrowright \frac{\tilde{X}}{N_1}$ and $\Lambda_0 \curvearrowright Y_0$ are conjugate through a non-singular isomorphism and a group isomorphism. But $\frac{\tilde{G}}{N} \curvearrowright \frac{\tilde{X}}{N}$ is weakly mixing and hence it is not an induced action by Lemma 1.8.

So we find $N_1 \lhd \tilde{G}$ open, such that $N_1 \curvearrowright \tilde{X}$ is proper and such that $\frac{\tilde{G}}{N_1} \curvearrowright \frac{\tilde{X}}{N_1}$ and $\frac{\tilde{G}}{N} \curvearrowright \frac{\tilde{X}}{N}$ are conjugate through the non-singular isomorphism $\Psi : \frac{\tilde{X}}{N_1} \to \frac{\tilde{X}}{N}$ and the group isomorphism $\delta : \frac{\tilde{G}}{N_1} \to \frac{\tilde{G}}{N}$. Furthermore we have $\Delta(N \cdot x) \in \frac{\tilde{G}}{N} \cdot \Psi(N_1 \cdot x)$ for almost all $x \in \tilde{X}$.

Set $P := (\mathbb{Z}[\frac{1}{p}] \times \mathbb{Z}_p)^n$. We first show that $N_1 \lhd P$. Let $q : \Gamma \to \mathrm{SL}_n(\mathbb{Z})$ be the quotient map as in Notation 2.5 and write $\Gamma_0 = \ker(q)$. Then $\Gamma_0 \curvearrowright \tilde{X}$ is given by $\mathrm{id} \times \Gamma_0 \curvearrowright \mathbb{R}^n \times (\mathbb{Z}_p^n)^\Gamma$ where Γ_0 acts on $(\mathbb{Z}_p^n)^\Gamma$ by Bernoulli shift. As $N_1 \cap \Gamma_0 \curvearrowright \tilde{X}$ is still essentially free and proper, this implies that $N_1 \cap \Gamma_0$ has to be finite as it is a Bernoulli action. But then $N_1 \cap \Gamma_0$ is trivial since it is normal in Γ_0.

Now take $xg \in N_1$ where $x \in P, g \in \Gamma$. Then for all $\lambda \in \Gamma_0$ we have

$$N_1 \ni (xg)^{-1}\lambda(xg)\lambda^{-1} = g^{-1}x^{-1}\lambda xg\lambda^{-1} = g^{-1}\lambda g\lambda^{-1} \in \Gamma_0 .$$

So $\lambda g \lambda^{-1} = g$ for all $\lambda \in \Gamma_0$, hence $g = e$. We have found that $N_1 \lhd P$.

As N_1 is open, there exists $l \in \mathbb{N}$ such that $(\{0\} \times p^l\mathbb{Z}_p)^n < N_1$. Since $(\mathbb{Z}[\frac{1}{p}] \times p^l\mathbb{Z}_p)^n < P$ has finite index, also $N_2 := N_1 \cap (\mathbb{Z}[\frac{1}{p}] \times p^l\mathbb{Z}_p)^n < N_1$

has finite index. It is clear that $N_2 = N_3 \times (p^l\mathbb{Z}_p)^n$ for some $N_3 < \mathbb{Z}[\frac{1}{p}]^n$. Furthermore N_3 is $\mathrm{SL}_n(\mathbb{Z})$-invariant as N_2 is normal in $\tilde{G}$.

By Lemma 4.8, N_3 is either $\{0\}$, $(a\mathbb{Z})^n$ for some $a \in \mathbb{Z}[\frac{1}{p}]$ or $(b\mathbb{Z}[\frac{1}{p}])^n$ for some $b \in \mathbb{Z}$. As there exists a non-singular conjugacy between the ergodic, measure preserving actions $\frac{\tilde{G}}{N} \curvearrowright (\frac{\tilde{X}}{N}, \mu_N)$ and $\frac{\tilde{G}}{N_1} \curvearrowright (\frac{\tilde{X}}{N_1}, \mu_{N_1})$ we can define a measure ν on $\frac{\tilde{X}}{N}$ by setting $\nu(A) := \mu_{N_1}(\Psi^{-1}(A))$. As Ψ is non-singular, ν and μ_N and $\frac{\tilde{X}}{N}$ are in the same measure class. By ergodicity of the actions, the Radon-Nikodym derivative is constant. Now $\frac{\tilde{X}}{N}$ has finite measure, and hence $N_1 \curvearrowright \tilde{X}$ has finite covolume. It follows that ψ is measure preserving. As N_2 has finite index in N_1, also $N_2 \curvearrowright \tilde{X}$ has finite covolume. So $N_3 \neq \{0\}$. Furthermore $N_2 \curvearrowright \tilde{X}$ is proper, hence $N_3 \neq (b\mathbb{Z}[\frac{1}{p}])^n$ for $b \in \mathbb{Z}$.

We conclude that $N_3 = (a\mathbb{Z})^n$ for some $a \in \mathbb{Z}[\frac{1}{p}]$ and $N_2 = (a\mathbb{Z} \times p^l\mathbb{Z}_p)^n$.

In particular we found $N_2 < N_1$ such that N_2 and $N = \mathbb{Z}^n \times \mathbb{Z}_p^n$ are commensurate. But as N_2 has finite index in N_1, this implies that N_1 and N are commensurate.

Since $\Delta(N \cdot x) \in \frac{\tilde{G}}{N} \cdot \Psi(N_1 \cdot x)$ for almost all $x \in \tilde{X}$, it is now clear that the compression constant $c(\Delta)$ equals the compression constant of the canonical stable orbit equivalence between $\frac{\tilde{G}}{N} \curvearrowright \frac{\tilde{X}}{N}$ and $\frac{\tilde{G}}{N_1} \curvearrowright \frac{\tilde{X}}{N_1}$. Hence $c(\Delta) = \frac{[N_1 : N \cap N_1]}{[N : N \cap N_1]}$.

Set

$$p_1 : N \hookrightarrow P \twoheadrightarrow \frac{P}{N_1} \hookrightarrow \frac{\tilde{G}}{N_1} \cong \frac{\tilde{G}}{N} \quad \text{and} \quad p_2 : N_1 \hookrightarrow P \twoheadrightarrow \frac{P}{N} \hookrightarrow \frac{\tilde{G}}{N} .$$

As N and N_1 are commensurate, both $\mathrm{Im}(p_1)$ and $\mathrm{Im}(p_2)$ are finite abelian normal subgroups of $\frac{\tilde{G}}{N}$. But then, denoting by p_Γ the projection onto Γ, $p_\Gamma(\mathrm{Im}(p_i))$ is trivial for $i = 1, 2$, so $\mathrm{Im}(p_i)$ is a finite subgroup of a p-group. One verifies that $|\,\mathrm{Im}(p_i)|$ is a power of p^n, say $|\,\mathrm{Im}(p_i)| = p^{nl_i}$ for $i = 1, 2$, $l_i \in \mathbb{Z}$. So we see that

$$\begin{aligned} p^{nl_1} &= |\,\mathrm{Im}(p_1)| = [N : \ker(p_1)] = [N : N \cap N_1] , \\ p^{nl_2} &= |\,\mathrm{Im}(p_2)| = [N_1 : \ker(p_2)] = [N_1 : N \cap N_1] . \end{aligned}$$

It follows that $c(\Delta) = \frac{[N_1 : N \cap N_1]}{[N : N \cap N_1]} = p^{kn}$ for some $k \in \mathbb{Z}$. Since $\mathcal{F}(\mathcal{R}_2)$ is the group generated by the compression constants of stable orbit equivalences between $\mathcal{R}_2$ and itself, we find that $\mathcal{F}(\mathcal{R}_2) \subset \{p^{kn} \mid k \in \mathbb{Z}\}$.

To prove the other inclusion remark that $N_1 = \frac{1}{p}\mathbb{Z}^n \times \mathbb{Z}_p^n \lhd \mathbb{Z}[\frac{1}{p}]^n \times \mathbb{Z}_p^n$ is open and that the restricted action $N_1 \curvearrowright \tilde{X}$ is proper (for the same reason $N \curvearrowright \tilde{X}$ was proper). Define the group isomorphism $\delta : \frac{\tilde{G}}{N_1} \to \frac{\tilde{G}}{N}$ by $\delta(z, s, g) = (pz, s, g)$. Set $\Psi : \frac{\tilde{X}}{N_1} \to \frac{\tilde{X}}{N} : \Psi(x, y) = (px, y)$. It is clear that this defines a measure preserving isomorphism. Then composition of ψ with the canonical stable orbit equivalence between $\frac{\tilde{G}}{N} \curvearrowright \frac{\tilde{X}}{N}$ and $\frac{\tilde{G}}{N_1} \curvearrowright \frac{\tilde{X}}{N_1}$ defines a self stable orbit equivalence of $\frac{\tilde{G}}{N} \curvearrowright \frac{\tilde{X}}{N}$ with compression constant p^n. So $\mathcal{F}(\mathcal{R}_2) = \{p^{kn} \mid k \in \mathbb{Z}\}$. $\square$

Chapter 5

Equivalence relations with ‘arbitrary’ fundamental group and McDuff II_1 factor

In this chapter, we construct examples of II_1 equivalence relations $\mathcal{R}$ that can have ‘arbitrary’ fundamental group, but such that nevertheless $\mathcal{L}(\mathcal{R})$ is McDuff (see Definition 1.34) and hence has $\mathbb{R}^*_+$ as fundamental group. This ‘arbitrary’ does not mean the fundamental group can be any subgroup of $\mathbb{R}^*_+$, but any group in a large class of groups $\mathcal{S}_{\mathrm{centr}}$ introduced by Popa and Vaes in [PV08a]. This large class contains all groups of the form $\exp(H)$ where H can be an uncountable group with any Hausdorff dimension $\alpha \in (0,1)$. Thus we show that the difference in fundamental group between a II_1 equivalence relation and its II_1 factor can be many things, as any group that is known to appear as the fundamental group of an equivalence relation, can appear while the II_1 factor has $\mathbb{R}^*_+$ as fundamental group.

First we prove a cocycle vanishing result. In Section 2 we use this to prove that in some situations the fundamental group of the orbit equivalence relation coming from a direct product of two actions is ‘the product of the fundamental groups of the orbit equivalence relations coming from each of the actions’. This is then applied to construct II_1 equivalence relations with ‘arbitrary’ fundamental group and McDuff II_1 factor, based on an example given by Popa in section 6.1 of [Po06].

5.1 A cocycle vanishing result

The result that we want to prove is a generalization of Theorem 9.1.1 in [Zi84]. There it is shown that a cocycle from a property (T) group to an amenable torsion free group has to be cohomologous to the identity cocycle. We will use a weaker group property, called the Haagerup property, and show that this result still holds.

Definition 5.1. (See also [CCJJV01]) Let Γ be a countable discrete group. We say that Γ has the Haagerup property if there exists a unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ such that

(1) $\forall \xi \in \mathcal{H} : \langle \pi(g)\xi, \xi \rangle \to 0$ as $g \to \infty$,

(2) there is a sequence $(\xi_n)_n$ of almost invariant unit vectors, so

$$\|\pi(g)\xi_n - \xi_n\|_2 \to 0.$$

Examples of groups with the Haagerup property are all amenable groups and $\mathbb{F}_n$ for any n.

Lemma 5.2. *Let Γ be a group with property (T), and $\Gamma \curvearrowright (X, \mu)$ a probability measure preserving action on the standard probability space (X, μ). Let $\omega : \Gamma \times X \to \Lambda$ be a cocycle, where Λ is a torsion free discrete group with the Haagerup property. Then ω is cohomologous to the identity cocycle.*

Proof. It is enough to prove the following claim:

Claim 5.3. *Given any non-negligible Γ-invariant subset U of X, there is a non-negligible Γ-invariant subset $V \subset U$ and a measurable function $\varphi : V \to \Lambda$ such that*

$$\omega(g, x) = \varphi(g \cdot x)\varphi(x)^{-1} \qquad \textit{for almost all } x \in V \textit{ and all } g \in \Gamma. \tag{5.1}$$

To see that this is indeed enough, assume the claim holds. Let

$$\begin{aligned} \Phi := \{((\varphi_i, V_i)_{i\in\mathcal{I}}, \mathcal{I}) \quad | \quad & \varphi_i \text{ have 2 by 2 disjoint domain and image,} \\ & \text{all } \varphi_i \text{ satisfy (5.1) for all } x \in V_i, \mu(V_i) \neq 0\}. \end{aligned}$$

We say $((\varphi_i, V_i)_{i\in\mathcal{I}}, \mathcal{I}) \preceq ((\varphi'_i, V'_i)_{i\in\mathcal{I}'}, \mathcal{I}')$ if $\mathcal{I} \subseteq \mathcal{I}'$, $V_i = V'_i$ for all $i \in \mathcal{I}$ and $\varphi_i(x) = \varphi'_i(x)$ for almost all $x \in V$ and all $i \in \mathcal{I}$. Then Φ is a partially ordered set. Furthermore any ordered chain in Φ clearly has an upper bound. So by Zorn's lemma there is a maximal element $((\varphi_i, V_i)_{i\in\mathcal{I}}, \mathcal{I})$ in Φ. But then $V := \sqcup_i V_i = X$ up to measure 0. Indeed, suppose $X\backslash V$ is non-negligible. Then, by the claim, there is a non-negligible subset $\tilde{V} \subset X\backslash V$ and a measurable function $\tilde{\varphi}$ satisfying the above properties. But then we could add $(\tilde{\varphi}, \tilde{V})$ to the family $((\varphi_i, V_i)_{i\in\mathcal{I}}, \mathcal{I})$, contradicting its maximality and thus proving the theorem.

Let $U \subset X$ be a non-negligible Γ-invariant subset of X. As Λ has the Haagerup property, let $\pi : \Lambda \to \mathcal{U}(\mathcal{H})$ be the representation from definition 5.1. Denote by $\mathcal{K} := \mathrm{L}^2(X, \mathcal{H})$ and consider the following unitary representation:

$$\theta : \Gamma \to \mathcal{U}(\mathcal{K}) \text{ where } (\theta(g^{-1})\xi)(x) = \pi(\omega(g,x)^{-1})\xi(g\cdot x).$$

We define a sequence $(\eta_n)_n \subset \mathcal{K}$ by $\eta_n(x) = \xi_n$, where ξ_n are the almost invariant vectors from Definition 5.1. We have for all $x \in X$ and all $g \in \Gamma$ that

$$\|\eta_n(x) - (\theta(g^{-1})\eta_n)(x)\|_{\mathcal{H}} = \|\xi_n - \pi(\omega(g,x)^{-1})\xi_n\|_{\mathcal{H}} \to 0 \tag{5.2}$$

as $n \to \infty$. Now since $\|\eta_n(x) - \theta(g^{-1})\eta_n(x)\|_{\mathcal{H}} \leq 2$, by the dominated convergence theorem it follows that $(\eta_n)_n$ is a sequence of almost invariant vectors. But since Γ has property (T), θ has a non-zero invariant vector $\eta \in \mathcal{K}$. Since we can choose η arbitrarily close to η_n for $n \to \infty$ (see Proposition 1.54), we can choose η such that on a non-negligible $V \subset U$, η is almost everywhere non-zero. Furthermore we have that for each $g \in \Gamma$,

$$((\pi\circ\omega)(g,x)\eta)(x) = \eta(g\cdot x).$$

Since $\langle\pi(g)\xi, \zeta\rangle \to 0$ as $g \to \infty$ for all $\xi, \zeta \in \mathcal{H}$, $\pi(g)\xi \to 0$ in the weak topology as $g \to \infty$ in Λ. But then each Λ-orbit in $\mathcal{H}$ is norm closed, so as $\mathcal{H}$ is separable, $\frac{\mathcal{H}}{\Lambda}$ is a well defined Polish space. Hence we can consider the map $\eta' : X \to \frac{\mathcal{H}}{\Lambda}$. Furthermore η' is Γ invariant, and hence by ergodicity it is constant. As η is almost everywhere non-zero on V, choose $\tilde{\eta} \in \mathcal{H}\backslash\{0\}$ such that $\eta'(x) = [\tilde{\eta}]$ for almost all $x \in X$. Let $\Lambda = \{\lambda_0, \lambda_1, \ldots\}$ be an enumeration of Λ. Let $V_i = \eta^{-1}(\pi(\lambda_i)\tilde{\eta})$. Define $\varphi : X \to \Lambda$ as

$$\varphi : X \to \Lambda : x \mapsto \begin{cases} \lambda_0 & \text{for } x \in V_0 \\ \lambda_1 & \text{for } x \in V_1 \ominus V_0 \\ \lambda_2 & \text{for } x \in V_2 \ominus (V_0 \oplus V_1) \\ \cdots \end{cases}$$

Then φ is a measurable map, and

$$\pi(\varphi(g\cdot x)^{-1}\omega(g,x)\varphi(x))\tilde{\eta}(x) = \tilde{\eta}(x).$$

Now since $\tilde{\eta} \neq 0$,

$$\varphi(g \cdot x)^{-1}\omega(g,x)\varphi(x) \in \mathrm{Stab}(\tilde{\eta}).$$

But the stabilizer of any non-zero element of $\mathcal{H}$ is finite, because $\pi(s)\xi \to 0$ in the weak-*-topology as $s \to \infty$. Since we assumed Λ to be torsion free, this proves the claim. $\square$

5.2 Equivalence relations with 'arbitrary' fundamental group and McDuff II_1 factor

We now use the Lemma proven in the previous section to show the following theorem. Recall that we call $\Delta : X \to Y$ a local isomorphism whenever there is a countable partition $X = \sqcup_i X_i$ into measurable subsets such that $\Delta_{|X_i}$ is a non-singular isomorphism between X_i and $\Delta(X_i)$.

Theorem 5.4. *Let Γ be a countable discrete group that has a subgroup Γ_0 with property (T). Let Λ be a countable discrete group with the Haagerup property that is torsion free. Let $(X,\mu),(Y,\nu)$ be standard probability spaces, and $\Gamma \overset{\alpha}{\curvearrowright} (X,\mu)$, $\Lambda \overset{\beta}{\curvearrowright} (Y,\nu)$ free, ergodic probability measure preserving actions, such that the restriction of α to Γ_0 is still ergodic. Assume that $\mathcal{F}(\mathcal{R}(\Gamma \curvearrowright X)) = F_1$ and $\mathcal{F}(\mathcal{R}(\Lambda \curvearrowright Y)) = F_2$. Then*

$$\mathcal{F}(\mathcal{R}(\Gamma \times \Lambda \curvearrowright X \times Y)) = \{st \mid s \in F_1, t \in F_2\} =: F_1F_2,$$

where $(g,s)\cdot(x,y) = (gx, sy)$.

Before we prove this, we first prove the following lemma.

Lemma 5.5. *Suppose $\Delta : X \times Y \to X \times Y$ and $\tilde{\Delta} : X \times Y \to X \times Y$ are local isomorphisms. Suppose the second component of Δ and $\tilde{\Delta}$ does not depend on the X-variable. Write*

$$\Delta(x,y) = (\Delta_{1,y}(x), \Delta_2(y)) \text{ and } \tilde{\Delta}(x,y) = (\tilde{\Delta}_{1,y}(x), \tilde{\Delta}_2(y)).$$

If $\Delta_2 \circ \tilde{\Delta}_2$ and $\tilde{\Delta}_2 \circ \Delta_2$ are local isomorphisms of Y, then Δ_2 and $\tilde{\Delta}_2$ are local isomorphisms of Y and $\Delta_{1,y}$ and $\tilde{\Delta}_{1,y}$ are local isomorphisms for almost all $y \in Y$.

Proof. Let $U \subset Y$ be a non-negligible subset of Y. Then $\Delta(X \times U) \subset X \times \Delta_2(U)$ is also non-negligible, so in particular $\nu(\Delta_2(U)) > 0$. The same reasoning applies to $\tilde{\Delta}_2$. As $\tilde{\Delta}_2 \circ \Delta_2$ is a local isomorphism, we can write $Y = \sqcup_i Y_i$ such that every $(\tilde{\Delta}_2 \circ \Delta_2)_{|Y_i}$ is a non-singular automorphism. The same arguments yields a partition $Y = \sqcup_j \tilde{Y_j}$ and $(\Delta_2 \circ \tilde{\Delta_2})_{|\tilde{Y_j}}$ which are all non-singular automorphisms. But then for every i, j the map $(\Delta_2)_{|Y_i \cap \tilde{\Delta_2}(\tilde{Y_j})}$ is a non-singular automorphism. In particular, Δ_2 is a local isomorphism of Y, and the same reasoning applies to $\tilde{\Delta}_2$.

To prove that almost all $\Delta_{1,y}$ are local isomorphisms of X (the reasoning for $\tilde{\Delta}_{1,y}$ is the same), take a non-negligible subset $V \subset X \times Y_i$ for some i such that $\Delta_{|V}$ is a non-singular isomorphism onto its image $W = \Delta(V)$. Set

$$V_y := \{x \in X \mid (x, y) \in V\}.$$

Let $f : W \to \mathbb{R}_+^*$ be the Radon-Nikodym derivative of $\Delta_{|V}$ and $g : \Delta_2(Y_i) \to \mathbb{R}_+^*$ the Radon-Nikodym derivative of $(\Delta_2)_{|Y_i}$. One can show that the Radon-Nikodym derivative of $(\Delta_{1,y})_{|V_y}$ is

$$h_y(x) = \frac{f(x, \Delta_2(y))}{g(\Delta_2(y))}.$$

Indeed, we have for all measurable $a : X \to \mathbb{R}, b : Y \to \mathbb{R}$ that

$$\begin{aligned}
&\int_{Y_i} b(\Delta_2(y)) \int_{V_y} a(\Delta_{1,y}(x)) d\mu d\nu \\
&= \int_V (a \otimes b)(\Delta(x, y)) d(\mu \times \nu) \\
&= \int_W (a \otimes b)(x, y) f(x, y) d(\mu \times \nu) \\
&= \int_{Y_i} b(\Delta_2(y)) \int_{W_{\Delta_2(y)}} a(x) \frac{f(x, \Delta_2(y))}{g(\Delta_2(y))} d\mu d\nu
\end{aligned}$$

In particular, $(\Delta_{1,y})_{V_y}$ is non-singular. As we assumed Δ_V was bijective, $(\Delta_{1,y})_{V_y}$ is a non-singular isomorphism onto its image, for almost all $y \in Y$ with $\mu(V_y) > 0$.

As we can partition $X \times Y$ into a countable disjoint union of sets like V, $\Delta_{1,y}$ is a local isomorphism for almost all $y \in Y$. $\square$

Proof of 5.4. Denote $\mathcal{R}(\Gamma \times \Lambda \curvearrowright X \times Y) =: \mathcal{R}$, $\mathcal{R}(\Gamma \curvearrowright X) =: \mathcal{R}_1$ and $\mathcal{R}(\Lambda \curvearrowright Y) =: \mathcal{R}_2$. Let

$$\Delta : X \times Y \to X \times Y : (x, y) \mapsto (\Delta_1(x, y), \Delta_2(x, y))$$

be a stable orbit equivalence for $\mathcal{R}$, i.e. for almost all $(x, y) \in X \times Y$ we have $\Delta((\Gamma \times \Lambda) \cdot (x, y)) = (\Gamma \times \Lambda) \cdot \Delta(x, y)$ and Δ is a stable automorphism of $X \times Y$. Let

$$\omega : \Gamma \times \Lambda \times X \times Y \to \Gamma \times \Lambda : (g, s, x, y) \mapsto (\omega_1(g, s, x, y), \omega_2(g, s, x, y))$$

be the associated Zimmer 1-cocycle, i.e.

$$\Delta_i(gx, sy) = \omega_i(g, s, x, y)\Delta_i(x, y).$$

Now let $\omega_2' : \Gamma_0 \times X \times Y \to \Lambda : (g, x, y) \mapsto \omega_2(g, e, x, y)$. One checks that this is a 1-cocycle taking values in Λ. Now Λ has the Haagerup property and Γ_0 has property (T), so by Theorem 5.2 ω_2' is cohomologous to the identity cocycle. So we find a measurable map $\varphi : X \times Y \to \Lambda$ such that

$$\omega_2'(g, x, y) = \varphi(g \cdot x, y)^{-1}\varphi(x, y) \text{ for all } g \in \Gamma_0.$$

Setting $\Delta_2'(x, y) = \varphi(x, y) \cdot \Delta_2(x, y)$, we have

$$\begin{aligned} \Delta_2'(gx, y) &= \varphi(gx, y)\Delta_2(gx, y) \\ &= \varphi(gx, y)\varphi(gx, y)^{-1}\varphi(x, y)\Delta_2(x, y) \\ &= \Delta_2'(x, y), \end{aligned}$$

for all $g \in \Gamma_0$. But the restriction of α to Γ_0 is still ergodic, so Δ_2' is essentially independent of x. Write $\Delta_2'(x, y) = \psi(y)$, where equality holds up to measure 0. We have

$$\Delta'(x, y) = (\Delta_{1,y}(x), \psi(y)).$$

We claim that ψ preserves $\mathcal{R}_2$ and is countable to one.

The same argument as above can be applied to Δ^{-1}, where Δ^{-1} is defined such that $\Delta^{-1}\Delta X_i = X_i$ for all i. Hence we obtain

$$\tilde{\Delta}(x, y) = (\tilde{\Delta}_{1,y}(x), \tilde{\psi}(y)).$$

Observe that $\psi(\Lambda \cdot y) \subset \Lambda \cdot \psi(y)$. Indeed, let $y \in Y$ and $s \in \Lambda$. Then

$$\begin{aligned} \Delta_2'(x, s \cdot y) &= \varphi(x, s \cdot y) \cdot \Delta_2(x, s \cdot y) \\ &= \varphi(x, s \cdot y) \cdot (\omega_2(e, s, x, y) \cdot \Delta_2(x, y)) \\ &= (\varphi(x, s \cdot y)\omega_2(e, s, x, y)\varphi(x, y)^{-1}) \cdot \Delta_2'(x, y). \end{aligned}$$

Similarly one shows that $\tilde{\psi}(\Lambda \cdot y) \subset \Lambda \cdot \tilde{\psi}(y)$. But then $\tilde{\psi}(\psi(y)) \in \Lambda \cdot y$ for almost all $y \in Y$. In particular, $\psi \circ \tilde{\psi}$ and $\tilde{\psi} \circ \psi$ are stable automorphisms of Y. By Lemma 5.5, we get that ψ and $\Delta_{1,y}$ are stable automorphisms of Y resp. X. In particular, they are both countable to one.

We now prove that ψ preserves $\mathcal{R}_2$, i.e. $y \sim_{\mathcal{R}_2} y'$ if and only if $\psi(y) \sim_{\mathcal{R}_2} \psi(y')$. We already showed that if $y \sim_{\mathcal{R}_2} y'$ then $\psi(y) \sim_{\mathcal{R}_2} \psi(y')$.

Now assume $\psi(y) \sim_{\mathcal{R}_2} \psi(y')$, then $\Delta_2(x,y) \sim_{\mathcal{R}_2} \Delta_2(x',y')$ for almost all $x, x' \in X$. We prove that for almost y, y' there exist non-negligible subsets $U, U' \subset X$ such that for all $x \in U$ there is $x' \in U'$ such that $\Delta(x,y) \sim_{\mathcal{R}} \Delta(x',y')$. Note that if this holds, then for almost all $y \in Y$ we find $\psi(y) \sim_{\mathcal{R}_2} \psi(y')$ implies $y \sim_{\mathcal{R}_2} y'$. As $\Delta_{1,y}$ is countable-to-one for almost all $y \in Y$, we find that $\mu(\Delta_{1,y}(X)) > 0$ for almost all $y \in Y$.

But then there is $g \in \Gamma$ such that $\mu(\Delta_{1,y}(X) \cap g \cdot \Delta_{1,y'}(X)) > 0$. So we find non-negligible subset U, U' such that for all $x \in U$ there is $x' \in U'$ such that $\Delta_1(x,y) \sim_{\mathcal{R}_1} \Delta_1(x',y')$.

Write $Y = \bigsqcup Y_n$ as a union of non-negligible sets Y_n such that the restriction of ψ to each Y_n is one-to-one. Now the scaling factor of each of these restriction has to be the same by ergodicity, say c. Clearly c is an element of F_2 as ψ is a stable orbit equivalence of $\mathcal{R}_2$.

By ergodicity, it suffices to restrict Δ to $X \times Y_i$ for some i. If we now replace Δ by

$$\Delta' = (\mathrm{Id} \times \psi)^{-1} \circ \Delta,$$

we can assume $\Delta'(x,y) = (\Delta_{1,y}(x), y)$, where $\Delta_{1,y}(x) = \Delta_1(x,y)$. The scaling factor of Δ' is clearly the product of the scaling factors of Δ (i.e. t) and $(\mathrm{Id} \times \psi)^{-1}$ (i.e. c^{-1}), so the scaling factor of Δ' is $c^{-1}t$.

But since we restricted our maps to $X \times Y_i$, Δ' is also a partial automorphism of $X \times Y$, that preserves the equivalence relation. Hence for almost all $y \in Y_i$, $\Delta_{1,y}$ preserves $\mathcal{R}_1$, so the scaling factor of $\Delta_{1,y}$ is an element of F_1 for almost all $y \in Y_i$. But this means that $c^{-1}t \in F_1$, so $t = cd$ for some $d \in F_1$. This proves that $\mathcal{F}(\mathcal{R}) = F_1 F_2$. □

Now using this, we can give examples of II_1 equivalence relations with any group in Popa and Vaes' $\mathcal{S}_{\mathrm{centr}}$ (see [PV08a]) as a fundamental group, and a McDuff II_1 factor. Before we do this, we briefly recall what $\mathcal{S}_{\mathrm{centr}}$ is.

$$\begin{aligned}\mathcal{S}_{\mathrm{centr}} = \{\mathcal{F} \subset \mathbb{R}^*_+ \mid \ &\text{there exists } \Lambda \curvearrowright (Y,\nu) \text{ a free, ergodic, m.p. action} \\ &\text{with } \Lambda \text{ amenable and } \operatorname{mod}(\mathrm{Centr}_\Lambda(Y)) = \mathcal{F}\}.\end{aligned}$$

In this definition, $\mathrm{Centr}_\Lambda(Y)$ are the automorphisms of Y that commute with the action of Λ, and for such an automorphism φ, $\mod(\varphi)$ is the scaling factor by which φ scales the infinite measure ν.

This class of groups contains $\exp(H)$ where $H < \mathbb{R}$ are subgroups of $\mathbb{R}$ which can have arbitrary Hausdorff dimension. In [PV08a], Popa and Vaes showed that any group in $\mathcal{S}_{\mathrm{centr}}$ can appear as the fundamental group of a II_1 factor and orbit equivalence relation, arising from a free ergodic measure preserving action of $\mathbb{F}_\infty$. These were the first examples of II_1 factors and equivalence relations with uncountable fundamental group different from $\mathbb{R}^*_+$.

Corollary 5.6. *Let $\Gamma \curvearrowright X$ be the example of Popa in [Po06, §6.1]. For any $F \in \mathcal{S}_{\mathrm{centr}}$, there is an action $\mathbb{F}_\infty \curvearrowright Y$ such that $\mathcal{F}(\mathcal{R}(\Gamma \times \mathbb{F}_\infty \curvearrowright X \times Y)) = F$ and $\mathcal{F}(\mathrm{L}^\infty(X \times Y) \rtimes (\Gamma \times \mathbb{F}_\infty)) = \mathbb{R}^*_+$.*

Proof. In section 6.1 of [Po06], Popa gave an example of a free ergodic probability measure preserving action $\Gamma \curvearrowright X$ such that Γ contains a property (T) subgroup Γ_0 that still acts ergodically, $\mathcal{F}(\mathcal{R}(\Gamma \curvearrowright X)) = \{1\}$ and $\mathcal{F}(\mathrm{L}^\infty(X) \rtimes \Gamma) = \mathbb{R}^*_+$. For the sake of completeness we recall the example.

Let Γ_0 be a countable property (T) group with no finite normal subgroups. Let $\Gamma_1 = \sum_n H_n$ be an infinite direct sum of non-abelian groups, again with no finite normal subgroups. Let $H_n \curvearrowright [0,1]$ be any free H_n action preserving the Lebesgue measure, and let $\Gamma_1 \curvearrowright [0,1]^{\mathbb{N}}$ be the product of these actions. Denote $X := ([0,1]^{\mathbb{N}})^{\Gamma_0}$ and let $\Gamma_0 \curvearrowright X$ by Bernoulli shifts and $\Gamma_1 \curvearrowright X$ diagonally, identically on each copy of $[0,1]^{\mathbb{N}}$. So we obtain an action

$$\Gamma_0 \times (\sum_n H_n) \curvearrowright ([0,1]^{\mathbb{N}})^{\Gamma_0}.$$

Since the Γ_0, Γ_1 actions commute they implement an action of $\Gamma = \Gamma_0 \times \Gamma_1 \curvearrowright (X, \mu)$ which is free. Since $\Gamma_0 \curvearrowright X$ is a Bernoulli action, $\Gamma \curvearrowright X$ is ergodic.

Any sequence of canonical unitaries v_{h_n} with $h_n \in H_n$ is central for $M := \mathrm{L}^\infty(X) \rtimes \Gamma$. By the non-commutativity of the H_n's it follows that these v_{h_n} are not hypercentral, so by [McD69, Theorem 3], $M \cong M \overline{\otimes} R$.

Now since Γ_0 is a property (T) group, the action $\Gamma_0 \curvearrowright X$ satisfies the conditions of Theorem 5.2 in [Po05], so it is cocycle superrigid. Since the action is also mixing and Γ_0 is normal in Γ, again by [Po05, Theorem 5.2] its extension to $\Gamma \curvearrowright X$ is cocycle superrigid. So by Theorem 5.7 in [Po05], $\mathcal{F}(\mathcal{R}_\Gamma)$ is countable. Since neither Γ_0 nor any of the H_n have a finite normal subgroup, it follows from [Po05, Theorem 5.8] that we have $\mathcal{F}(\mathcal{R}_\Gamma) = \{1\}$.

On the other hand, from the results in [PV08a] it follows that for any $F \in \mathcal{S}_{\text{centr}}$ there exists an action $\mathbb{F}_\infty \curvearrowright Y$ such that

$$\mathcal{F}(\mathcal{R}(\mathbb{F}_\infty \curvearrowright Y)) = \mathcal{F}(\mathrm{L}^\infty(Y) \rtimes \mathbb{F}_\infty) = F.$$

As $\mathbb{F}_\infty$ has the Haagerup property, we can use Theorem 5.4 to conclude that for any $F \in \mathcal{S}_{\text{centr}}$ there exists an action $\mathbb{F}_\infty \curvearrowright Y$ such that

$$\mathcal{F}(\mathcal{R}(\Gamma \times \mathbb{F}_\infty \curvearrowright X \times Y)) = F \text{ and } \mathcal{F}(\mathrm{L}^\infty(X \times Y) \rtimes (\Gamma \times \mathbb{F}_\infty)) = \mathbb{R}^+.$$

To see this last equality, it is clear that this II_1 factor is still a McDuff type factor, and hence $\mathcal{F}(\mathrm{L}^\infty(X \times Y) \rtimes (\Gamma \times \mathbb{F}_\infty)) = \mathbb{R}^*_+$. $\square$

Chapter 6

Conclusion

In this thesis, we first gave a setting in which a II_1 factor resulting from a group action admits a second Cartan subalgebra which can be non-conjugate to the original Cartan subalgebra. We used this setting to give an example of a II_1 factor with trivial fundamental group, but containing a second Cartan subalgebra such that the associated equivalence relation has non-trivial fundamental group. This was not absurd, as the second Cartan subalgebra was twisted by a 2-cocycle.

In the fifth chapter, we also gave an example of a group action, such that the associated equivalence relation could have any group in $\mathcal{S}_{\mathrm{centr}}$ as a fundamental group, while the associated II_1 factor was a McDuff II_1 factor, hence having $\mathbb{R}_+^*$ as fundamental group. From this, one can see that a priori there seems to be no restriction on the possibilities concerning differences in fundamental group between an equivalence relation and its associated II_1 factor.

There is however more that could be done. First of all, the construction of the example we study in chapters three and four may seem a little artificial. Originally, the example had a simpler setup, only involving $\mathrm{SL}_n(\mathbb{Z})$ instead of the amalgamated free product $\mathrm{SL}_n(\mathbb{Z}) *_\Sigma (\Sigma \times \Lambda)$. However, in this case, it was not clear whether partial automorphisms of the II_1 factor M embedded $\mathrm{L}^\infty(X) \rtimes \mathbb{Z}^n$ into itself (and the other way around). This was a crucial ingredient in the description of partial automorphisms of M. I am however convinced that the same results hold for the case where we use $\mathrm{SL}_n(\mathbb{Z})$ instead of the amalgamated free product. The proof would be the same, except that one first would need to prove that this embedding still holds.

More importantly, and more interesting to the theory in general, would be to obtain examples of group actions or equivalence relations such that the (orbit) equivalence relation has trivial fundamental group, whereas the associated II_1 factor can have non-trivial fundamental group. As to finding results of this sort, I have no intuition how they could be obtained. It is clear that this is a difficult problem: in most cases the fundamental group of the II_1 factor is calculated by showing that it is exactly the fundamental group of the underlying equivalence relation. In trying to obtain such examples using profinite actions (see chapter 2) I had no success, but I think it would be very interesting to expand on this. Ultimately, I think that in the non-twisted case the fundamental group of the equivalence relation holds no restriction on the fundamental group of the II_1 factor, other than that it has to be a subgroup.

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